1. Basic set definitions. Given sets A and B, and a universe U that contains all sets in question, we define:

(a) $A \cup B = \{x \in U : x \in A \text{ or } x \in B\}$.

(b) $A \cap B = \{x \in U : x \in A \text{ and } x \in B\}$.

(c) $A - B = \{x \in A : x \notin B\}$.

(d) $A \times B = \{\text{ordered pairs } (x, y) : x \in A \text{ and } y \in B\}$.

(e) $\overline{A} = U - A$.

(f) $\mathscr{P}(A) = \{\text{all subsets of } A\}$.

(g) $\left|P(A)\right| = 2^{|A|}$ for any set A.

(h) The statement $A \subseteq B$ is equivalent to the statement $x \in A \Rightarrow x \in B$.

---

2. Intersection and union of indexed sets. Given an indexing set I and a set A_α for each $\alpha \in I$, and a universe U, we define

(a) $\bigcup_{\alpha \in I} A_\alpha = \{x \in U : x \in A_\alpha \text{ for some } \alpha \in I\}$.

(b) $\bigcap_{\alpha \in I} A_\alpha = \{x \in U : x \in A_\alpha \text{ for all } \alpha \in I\}$.

---

3. Proof templates.

(a) $P \Rightarrow Q$, direct proof.

> **Theorem.** $P \Rightarrow Q$.
> **Proof.** Assume P. [Now do what you need to conclude:] Therefore, Q.
> So $P \Rightarrow Q$. $\square$

(b) $P \Rightarrow Q$, contrapositive proof.

> **Theorem.** $P \Rightarrow Q$.
> **Proof.** Assume $\sim Q$. [Now do what you need to conclude:] Therefore, $\sim P$.
> So $P \Rightarrow Q$. $\square$

(c) $P \Leftrightarrow Q$.

> **Theorem.** $P \Leftrightarrow Q$.
> **Proof.** Assume P. [Now do what you need to conclude:] Therefore, Q.
> So $P \Rightarrow Q$.
> Next, assume Q. [Now do what you need to conclude:] Therefore, P.
> So $Q \Rightarrow P$.
> Therefore, $P \Leftrightarrow Q$. $\square$

(d) $A \subseteq B$.

Theorem. $A \subseteq B$.
Proof. Assume $x \in A$. [Now do what you need to conclude:] Therefore, $x \in B$.
So $A \subseteq B$. $\square$

(e) $A = B$.

Theorem. $A = B$.
Proof. Assume $x \in A$. [Now do what you need to conclude:] Therefore, $x \in B$.
So $A \subseteq B$.
Now assume $x \in B$. [Now do what you need to conclude:] Therefore, $x \in A$.
So $B \subseteq A$.
Therefore, $A = B$. $\square$

(f) Proof by counterexample. To prove that a statement is false, you need only find one instance where the statement fails.

---

4. Some special sets.

(a) $\mathbb{Z} = \{\text{integers}\} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

(b) $\mathbb{N} = \{\text{natural numbers}\} = \{1, 2, 3, \ldots\}$.

(c) $\mathbb{R} = \{\text{real numbers}\} = (-\infty, \infty)$.

(d) $\mathbb{Q} = \{\text{rational numbers}\} = \{\text{fractions } m/n : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$.

(e) Let $a, b \in \mathbb{Z}$. We write $a + b\mathbb{Z}$ for the set $\{a + bm : m \in \mathbb{Z}\}$.

---

5. Facts about integers.

(a) Let $a, b \in \mathbb{Z}$. We say a divides b, written $a|b$, if $b = na$ for some $n \in \mathbb{Z}$.

(b) (Division algorithm.) Given integers a and b with $b > 0$, there exist unique integers q and r for which $a = qb + r$ and $0 \leq r < b$.