The binomial distribution.

A <u>binomial experiment</u> is one with only two possible outcomes, called a <u>success</u> and a failure.

<u>Example 1</u>

(a) Flip a fair coin; define success = coin lands heads. Then $P(\text{success}) = 0.5 = 50\%$.

(b) A 40% three-point shooter takes a 3-point shot; define success = shot is made. Then $P(\text{success}) = 40\%$.

(c) Roll a fair die: define success = die lands on 5. Then $P(\text{success}) = 1/6 \approx 16.67\%$.

<u>Proposition</u>

In a binomial experiment with $P(\text{success}) = p$, let X = number of successes (in a single trial). Then $E(X) = p$.

<u>Proof</u> For such an experiment, we have

$$E(X) = \sum_x x \cdot P(X = x)$$

$$= 0 \cdot P(X=0) + 1 \cdot P(X=1),$$

since the outcome is either a failure $(X=0)$ or a success $(X=1)$. Now $P(X=1) = P(\text{success}) = p$, so $P(X=0) = P(\text{failure}) = 1-p$ (since $P(\text{success}) + P(\text{failure})$ must equal 1), so

$$E(X) = 0 \cdot (1-p) + 1 \cdot p = p. \qquad \text{Д}$$

Example 2

If X is the number of 3-point field goals made by a 40% 3-point shooter in a single shot, then $E(X) = 0.4$.

Let's ramp it up a bit:

Example 3

A 40% 3-point shooter makes 7 attempts. Let X be the number made out of the 7. Find $P(X=4)$. Assume the shots are independent.

Solution.

Of the 7 shots, there are $\binom{7}{4}$ possibilities for which 4 go in.

For each of these $\binom{7}{4}$ possibilities, the probability of exactly these 4 being made, and the other 3 being missed, is, by Axiom 2 of last time,

$$\underbrace{0.4 \cdot 0.4 \cdot 0.4 \cdot 0.4}_{4 \text{ hits}} \cdot \underbrace{0.6 \cdot 0.6 \cdot 0.6}_{3 \text{ misses}} = 0.4^4 \, 0.6^3.$$

So, finally:

$$P(X=4) = \binom{7}{4} 0.4^4 \, 0.6^3$$
$$= 0.1935 = 19.35\%.$$

Let's ramp up even further:

Theorem.

If a binomial experiment, with P(success) = p, is repeated n times, and X is the number of successes in these n trials, then

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad (0 \le k \le n).$$

(assuming the trials are independent).
(Proof omitted: It's like Example 2 above.)

Example 4.

An unfair coin, with P(heads) = 1/3, is flipped 3 times. Let Y be the number of heads that come up. Find the pmf for Y. Also find E(Y).

Solution.

By the theorem,

$$P(Y=0) = \binom{3}{0}\cdot\left(\tfrac{1}{3}\right)^0\cdot\left(\tfrac{2}{3}\right)^3 = 8/27$$
$$P(Y=1) = \binom{3}{1}(1/3)^1(2/3)^2 = 12/27$$
$$P(Y=2) = \binom{3}{2}(1/3)^2(2/3)^1 = 6/27$$
$$P(Y=3) = \binom{3}{3}(1/3)^3(2/3)^0 = 1/27$$

(Note that the probabilities add to 1.)

Also,

$$E(Y) = 0\cdot 8/27 + 1\cdot 12/27 + 2\cdot 6/27 + 3\cdot 1/27$$
$$= 27/27 = 1.$$