More probability.

Recall: a random variable (rv) X assigns numbers to outcomes of an experiment. The probability mass function (pmf) for X is the function $p(X=x)$, as x runs through all possible values of X.

Example 1.

Flip three fair coins; let X be the number of heads showing. Find the pmf for X.

Solution.

Write the sample space S as

$$S = \{ \overbrace{HHH}^{X=3}, \underbrace{HHT, HTH, THH}_{X=2}, \overbrace{TTH, THT, HTT}^{X=1}, \underbrace{TTT}_{X=0} \}.$$

By Axiom 1 of last time, we compute:

$$P(X=0) = 1/8 = 0.125, \qquad P(X=1) = 3/8 = 0.375,$$
$$P(X=2) = 3/8 = 0.375, \qquad P(X=3) = 1/8 = 0.375$$

New definition: Given an rv X, we define the <u>expected value</u> $E(X)$ by

$$E(X) = \sum_x x \cdot P(X=x),$$

the sum being over all possible values x of X.

Example 2.

What is $E(X)$ for X as in Example 1?

Solution.

$$E(X) = \sum_x x \cdot P(X=x)$$

$$= 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3 \cdot P(X=3)$$

$$= 0 \cdot 1/8 + 1 \cdot 3/8 + 2 \cdot 3/8 + 3 \cdot 1/8 = \frac{0+3+6+3}{8} = \frac{12}{8}$$

$$= 1.5.$$

Remark: $E(X)$ is "a weighted sum of the values of X, with each value weighted by its probability of occurring." So, $E(X)$ is "what we expect X to be on average."

Example 3.

Roll two fair dice; let Y = the sum of the two numbers showing. Find $E(Y)$.

Solution.

Last time, we saw that

$$P(Y=y) = \frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36},$$

for $y = 2, 3, 4, 5, \ldots, 11, 12$ respectively.

So

$$E(Y) = \sum_y y \cdot P(Y=y)$$

$$= 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + \ldots + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36}$$

$= 7.$

Finally, for today:

Probability Axiom #2.

Suppose A and B are events; let AB be the event where both A and B occur. (AB is the same as $A \cap B$.) If A and B are independent (whether one occurs does not influence whether the other does), then

$$P(AB) = P(A)\,P(B).$$

Example 4.

Roll two fair dice. Find the probability that the first die lands on an even number and the second lands on a multiple of 3.

Solution. Assuming the dice are rolled independently, Axiom 2 says

$$\begin{aligned} &P(\text{first is even and second is a multiple of } 3) \\ &= P(\text{first is even}) \cdot P(\text{second is a multiple of } 3) \\ &= \tfrac{1}{2} \cdot \tfrac{1}{3} = \tfrac{1}{6} = 16.67\%. \end{aligned}$$

Remark: we could also count: the event in question is the set

$$\{23, 43, 63, 26, 46, 66\},$$

so the probability of this event is $\frac{6}{36} = \frac{1}{6} = 16.67\%.$