The binomial theorem, continued.

Lemma. Let $k \in \mathbb{N}$. For $1 \le j \le k$, we have

$$\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}.$$

Proof by counting.

Since the set
$$S = \{0, 1, 2, \ldots, k\}$$
has $k+1$ elements, we know, by definition of $\binom{k+1}{j}$, that S has $\binom{k+1}{j}$ subsets of size-j. If we can show that the number of size-j subsets of S is $\binom{k}{j} + \binom{k}{j-1}$, we'll be done.

But any size-j subset of S is of exactly one of the following types:

(a) Subsets that don't contain 0. Here, we must choose all j elements from the k elements not equal to 0. So there are $\binom{k}{j}$ such subsets.

(b) Subsets that contain 0. Once 0 is chosen, $j-1$ elements must be chosen from the remaining k elements of S. So there are $\binom{k}{j-1}$ subsets of this type.

By (a) and (b), $|S| = \binom{k}{j} + \binom{k}{j-1}$, and we're done. $\square$

Now we prove:

<u>The Binomial Theorem</u>. For $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

$$= \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j.$$

<u>SKETCH of proof</u> (by induction).

Let $A(n)$ be the given statement.

Step 1. Is $A(1)$ true?

$$(a+b)^1 \stackrel{?}{=} \binom{1}{0}a^1 + \binom{1}{1}b^1$$

$$a+b = a+b \checkmark$$

So $A(1)$ is true.

<u>Step 2</u> Rather than <u>proving</u> that $A(k) \Rightarrow A(k+1)$, let's <u>illustrate</u> it by showing that $A(4) \Rightarrow A(5)$.

$A(4)$ says:

$$(a+b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4.$$

It follows that

$$(a+b)^5 = (a+b)(a+b)^4 = a(a+b)^4 + b(a+b)^4$$

$$= \binom{4}{0}a^5 + \binom{4}{1}a^4b + \binom{4}{2}a^3b^2 + \binom{4}{3}a^2b^3 + \binom{4}{4}ab^4$$

$$+ \binom{4}{0}a^4b + \binom{4}{1}a^3b^2 + \binom{4}{2}a^2b^3 + \binom{4}{3}ab^4 + \binom{4}{4}ab^5$$

$$= \binom{4}{0}a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}ab^4 + \binom{4}{4}ab^5$$

(we added using the lemma).

Since $\binom{4}{0} = \binom{5}{0}$ and $\binom{4}{4} = \binom{5}{5}$ (all of these equal one), the above sum equals

$$\binom{5}{0}a^5 + \binom{5}{1}a^4 b + \binom{5}{2}a^3 b^2 + \binom{5}{3}a^2 b^3 + \binom{5}{4}ab^4 + \binom{5}{5}b^5,$$

so $A(5)$ follows. $\square$