The binomial theorem.

Question: how do we expand out $(a+b)^n$??

Answer: first look for a pattern:

$$(a+b)^1 = a+b$$
$$(a+b)^2 = (a+b)(a+b) = a^2+2ab+b^2$$
$$(a+b)^3 = (a+b)(a^2+2ab+b^2) = a(a^2+2ab+b^2)$$
$$+ b(a^2+2ab+b^2) = a^3+2a^2b+ab^2+a^2b+2ab^2+b^3$$
$$= a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = (a+b)(a^3+3a^2b+3ab^2+b^3)$$
$$= a(a^3+3a^2b+3ab^2+b^3)$$
$$+ b(a^3+3a^2b+3ab^2+b^3)$$
$$= a^4+3a^3b+3a^2b^2+ab^3$$
$$+ \quad a^3b+3a^2b^2+3ab^3+b^3$$
$$= a^4+4a^3b+6a^2b^2+4ab^3+b^3$$

The pattern is:

$$\boxed{\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 \\ &\quad + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}ab^n \\ &= \sum_{j=0}^{n}\binom{n}{j}a^{n-j}b^j.\end{aligned}}$$

The Binomial Theorem

For example,

$$(a+b)^6 = \binom{6}{0}a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3$$
$$+ \binom{6}{4}a^2b^4 + \binom{6}{5}ab^5 + \binom{6}{6}b^6$$

$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

To prove the binomial theorem, we'll need

<u>Lemma</u>. For $1 \le j \le k$, we have
$$\binom{k}{j-1} + \binom{k}{j} = \binom{k+1}{j}.$$

<u>Proof</u> (method 1: algebra).
By definition of binomial coefficients,

$$\binom{k}{j-1} + \binom{k}{j} = \frac{k!}{(j-1)!\,(k-j+1)!} + \frac{k!}{j!\,(k-j)!}.$$

To get a common denominator on the right, multiply the first fraction by j/j, and the second by $(k-j+1)/(k-j+1)$. Since $j\cdot(j-1)! = j!$ and $(k-j+1)\cdot(k-j)! = (k-j+1)!$, we get

$$\binom{k}{j-1} + \binom{k}{j} = \frac{j\cdot k!}{j!\,(k-j+1)!} + \frac{(k-j+1)\cdot k!}{j!\,(k-j+1)!}$$

$$= \frac{(j+k-j+1)\cdot k!}{j!\,(k-j+1)!}$$

$$= \frac{(k+1)\cdot k!}{j!\,(k-j+1)!} = \frac{(k+1)!}{j!\,(k-j+1)!}$$

$$= \binom{k+1}{j},$$

as promised. $\square$

Next time:

(a) a different proof of the lemma;
(b) A proof of the Binomial Theorem.