Counting sets, continued.

Recall: there are

$$\binom{n}{k}^{*} = \frac{n!}{k!(n-k)!}$$

different k-element subsets of a set with n elements.*

* "n choose k," also written ${}_nC_k$ or $C(n,k)$.

* Such a subset is called a k-combination of n.

Examples:

1) (Compare w/ the example of 4/3.) Given a standard 52-card deck, find the number of 5-card hands:

(0) in total;
(A) that are all of the same suit;
(B) with exactly one 3;
(C) with no 3's;
(D) with at least one 3;
(E) that are all of the same suit or have no 3's (or both).
(F) that are full houses (3 cards of one face value; 2 cards of another).

Solution.

(0) $\binom{52}{5} = \dfrac{52!}{5!\,47!} = \dfrac{52\cdot 51\cdot \overset{5}{\cancel{50}}\cdot 49\cdot \overset{4}{\cancel{48}}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}$

$= 52\cdot 51\cdot 5\cdot 49\cdot 4 = 2{,}598{,}960.$

(A) Method one: for each of the four suits, there are $\binom{13}{5}$ hands all of that suit. So in total, there are

$$4\binom{13}{5} = 5{,}148 \text{ such hands.}$$

Method two: we saw on 4/3 that there are 617,760 5-card lists of all one suit. So the number of such hands is

$$\frac{617{,}760}{5!} = 5{,}148.$$

(B) There are $\binom{4}{1}$ ways of choosing the 3, and $\binom{48}{4}$ ways of choosing ~~the~~ other cards, yielding

$$\binom{48}{4}\binom{4}{1} = 778{,}320 \text{ hands w/exactly}$$

one 3.

(D14: use another method.)

(C) There are

$$\binom{48}{5} = 1{,}712{,}304$$

hands where no card is a 3.

(D) Subtract the answer to (C) from the answer to (0):

$$2{,}598{,}960 - 1{,}712{,}304 = 886{,}656.$$

(E) The number of hands that are both of the same suit and have no 3's is $4\binom{12}{5} = 3{,}168.$

So the number with either (or both) of these properties is, by the inclusion-exclusion principle and parts (A) and (C) above,

$$5148 + 1712304 - 3168 = 1{,}714{,}284.$$

(F) Choose one face value, then choose 3 cards of that face value, then choose another face value, then choose 2 cards of that value. Count:

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3744.$$

Example 2 (tiny probability lesson): What's the probability of getting a full house when drawing 5-cards out of 52?

Solution

Divide the # of possible full houses by the total # of possible hands:

$$\frac{3744}{2598960} \approx 0.0014 \approx 0.14\%.$$