More counting.

A) Factorials.

Recall that, for $n \in \mathbb{N}$, we define $n!$ ("n factorial") by

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1.$$

E.g.

$$4! = 4\cdot 3\cdot 2\cdot 1 = 24,$$
$$10! = 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 3{,}628{,}800,$$
$$1! = 1.$$

We also define $0! = 1$.

Note that the number of k-permutations (that is, length-k lists, without repetition) of n objects, given by

$${}_nP_k = P(n,k) = n(n-1)(n-2)\cdots(n-k+1), \qquad (*)$$

can be written more compactly, if we multiply top and bottom of $(*)$ by $(n-k)!$:

$$P(n,k) = \frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)!}{(n-k)!}$$
$$= \frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)(n-k-1)\cdots 2\cdot 1}{(n-k)!}$$
$$= \frac{n!}{(n-k)!}. \qquad (**)$$

Example:

(a) Write $P(57,32)$ in terms of factorials.

(b) Evaluate $\frac{100!}{98!}$ without a calculator.

Solution.

(a) $P(57,32) = \dfrac{57!}{(57-32)!} = \dfrac{57!}{25!}$

(b) $\dfrac{100!}{98!} = 100 \cdot 99 = 9900.$

Remark: Formula (**) is easier to write down but, generally, formula (*) is easier to compute with.

B) Counting sets.

Question: how many size-k sets (order doesn't matter) can be made from n objects (without repetition)?

Answer: we know that $\frac{n!}{(n-k)!}$ size-k lists (without repetition) can be made from these objects. Each such list can be arranged in

$$k(k-1)(k-2)\cdots 2\cdot 1 = k!$$

ways. All these ways yield the same set. So:

FACT. Let $\binom{n}{k}$ ("n choose k") (also written ${}_nC_k$) denote the number of size k subsets of a set of size n. Then

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (0 \le k \le n).$$

Example: some basic computations.

(a) Express as natural numbers:

(i) $\binom{8}{4}$, (ii) $\binom{100}{97}$ (iii) $\binom{100}{3}$ (iv) $\binom{17}{17}$

(b) Explain why $\binom{n}{k} = \binom{n}{n-k}$ for $0 \le k \le n$, in two ways.

Solution.

(a) (i) $\binom{8}{4} = \frac{8!}{4!4!} = \frac{\cancelto{2}{8}\cdot 7\cdot \cancel{6}\cdot 5}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 2\cdot 7\cdot 5 = 70.$

(ii) $\binom{100}{97} = \frac{100!}{97!\,3!} = \frac{\cancelto{50}{100}\cdot\cancelto{33}{99}\cdot 98}{\cancel{3}\cdot\cancel{2}\cdot 1} = 50\cdot 33\cdot 98$

$= 161{,}700.$

(iii) $\binom{100}{3} = \frac{100!}{3!\,97!} \overset{\text{same as (ii)}}{=} 161{,}700.$

(iv) $\binom{17}{17} = \frac{\cancel{17!}}{\cancel{17!}\,0!} = \frac{1}{1} = 1.$

(b) First argument: by the FACT,

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{(n-k)!\,k!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}.$$

Second argument: given n items, including k of these in a set is the same as excluding $n-k$ of them. So, by counting these ways,

$$\binom{n}{k} = \binom{n}{n-k}.$$