## SOLUTIONS TO S-POP Exercise B(iv)-3

**Exercise B(iv)-3.** Show that

$$\lim_{n\to\infty} \frac{4n^3 + n + \sin n}{7n^3 + 3} = \frac{4}{7}.$$

**Solution.** Let $\varepsilon > 0$. [Scratchwork: We want $n$ large enough that

$$\left|\frac{4n^3 + n + \sin n}{7n^3 + 3} - \frac{4}{7}\right| < \varepsilon.$$

But, getting a common denominator and using the triangle inequality, we have

$$\begin{aligned}\left|\frac{4n^3 + n + \sin n}{7n^3 + 3} - \frac{4}{7}\right| &= \left|\frac{7(4n^3 + n + \sin n) - 4(7n^3 + 3)}{7(7n^3 + 3)}\right| = \left|\frac{7n + 7\sin n - 12}{7(7n^3 + 3)}\right| \\ &\le \frac{|7n| + |7\sin n| + |-12|}{7(7n^3 + 3)} \le \frac{7n + 7 + 12}{7(7n^3 + 3)} = \frac{7n + 19}{7(7n^3 + 3)}.\end{aligned}$$

(We used the fact that $|\sin n| \le 1$ always.) Now $19 \le 19n$, and $7n^3 + 3 > 7n^3$. So we get

$$\left|\frac{4n^3 + n + \sin n}{7n^3 + 3} - \frac{4}{7}\right| < \frac{7n + 19n}{7(7n^3)} = \frac{26n}{49n^3} = \frac{26}{49n^2}.$$

Solving $26/(49n^2) < \varepsilon$ for $n$ gives us $1/n^2 < 49\varepsilon/26$, or $n^2 > 26/(49\varepsilon)$, or $n > \sqrt{26/(49\varepsilon)}$. So this is what we write.] Let $R = \sqrt{26/(49\varepsilon)}$. Then $n > R \Rightarrow$

$$\begin{aligned}\left|\frac{4n^3 + n + \sin n}{7n^3 + 3} - \frac{4}{7}\right| &= \left|\frac{7(4n^3 + n + \sin n) - 4(7n^3 + 3)}{7(7n^3 + 3)}\right| = \left|\frac{7n + 7\sin n - 12}{7(7n^3 + 3)}\right| \\ &\le \frac{|7n| + |7\sin n| + |-12|}{7(7n^3 + 3)} \le \frac{7n + 7 + 12}{7(7n^3 + 3)} = \frac{7n + 19}{7(7n^3 + 3)} \\ &< \frac{7n + 19n}{7(7n^3)} = \frac{26n}{49n^3} = \frac{26}{49n^2} < \frac{26}{49R^2} \\ &= \frac{26}{49\left(\sqrt{26/(49\varepsilon)}\right)^2} = \frac{26}{49\left(26/(49\varepsilon)\right)} = \frac{1}{1/\varepsilon} = \varepsilon.\end{aligned}$$

So

$$\lim_{n\to\infty} \frac{4n^3 + n + \sin n}{7n^3 + 3} = \frac{4}{7}. \qquad \square$$