Limits, continued.

Recall:

Definition.

We say

$$\lim_{n\to\infty} x_n = L$$

if, $\forall \varepsilon > 0$, $\exists R \in \mathbb{R}$:

$$n > R \Rightarrow |x_n - L| < \varepsilon.$$

Strategy for limit proofs:

Let $\varepsilon > 0$.

[Scratchwork:

Start with the inequality $|x_n - L| < \varepsilon$.

Manipulate it (do algebra*) to turn it into an inequality of the form $n >$ something (dependent on ε). Now write this:]

Let $R =$ something. Then

$n > R \Rightarrow$ (now do the "reverse" algebra to show) that $|x_n - L| < \varepsilon$.

So $\lim_{n\to\infty} x_n = L$.

ATWMR.

*The following sometimes helps with the algebra:

<u>Lemma</u> (meaning "proposition (or theorem) helper"): the <u>triangle inequality</u> on $\mathbb{R}$.

$\forall x, y \in \mathbb{R}$:

$$|x|+|y| \le |x+y|. \qquad (\Delta_=)$$

(Proof omitted.)

We use this to help prove:

Proposition.

$$\lim_{n\to\infty} \frac{n^2+(-1)^n}{2n^2+10} = \frac{1}{2}.$$

<u>Proof</u>

Let $\varepsilon > 0$.

[Scratchwork: we want

$$\left| \frac{n^2+(-1)^n}{2n^2+10} - \frac{1}{2} \right| < \varepsilon. \qquad \text{Now}$$

$$\left| \frac{n^2+(-1)^n}{2n^2+10} - \frac{1}{2} \right| = \left| \frac{2(n^2+(-1)^n) - (2n^2+10)}{2(2n^2+10)} \right|$$

$$= \left| \frac{2n^2+2(-1)^n-2n^2-10}{4n^2+20} \right| = \left| \frac{2(-1)^n-10}{4n^2+20} \right|$$

$$\underset{\text{by } (\Delta_=)}{\le} \frac{|2(-1)^n|+|10|}{(4n^2+20)} = \frac{2+10}{4n^2+20} = \frac{12}{4n^2+20} = \frac{3}{n^2+5}$$

$$< \frac{3}{n^2}{}^{*}, \text{ since } n^2+5 > n^2.$$

We solve

$$\frac{3}{n^2} < \varepsilon$$

for n:

$$\frac{n^2}{3} > \frac{1}{\varepsilon}$$

$$n^2 > 3/\varepsilon$$

$n > \sqrt{3/\varepsilon}$. Now write this:]

Let $R = \sqrt{3/\varepsilon}$. Then

$$n > R \Rightarrow \left| \frac{n^2 + (-1)^n}{2n^2+10} - \frac{1}{2} \right| \leq \left| \frac{2(n^2+(-1)^n) - (2n^2+10)}{2(2n^2+10)} \right|$$

$$= \left| \frac{2(-1)^n - 10}{4n^2+20} \right| \leq \frac{12}{4n^2+20} = \frac{3}{n^2+5} < \frac{3}{n^2} < \frac{3}{R^2}$$

$$= \frac{3}{(\sqrt{3/\varepsilon})^2} = \frac{3}{3/\varepsilon} = \varepsilon .$$

So $\lim_{n\to\infty} \frac{n^2+(-1)^n}{2n^2+10} = \frac{1}{2}$. ATWMR

*Often, a helpful strategy is to reduce $|x_n - L| < \varepsilon$ to something of the form $\frac{\text{constant}}{\text{power of } n} < \varepsilon$.

This is easy to solve for n.