Limits of sequences.

Definition. Let

$$x_1, x_2, x_3, \ldots$$

be a sequence of real numbers. We say

$$\lim_{n\to\infty} x_n = L$$

if, $\forall \varepsilon > 0$, $\exists R \in \mathbb{R}$: $\overset{*}{\overline{n > R}} \Rightarrow \overset{*}{\overline{|x_n - L| < \varepsilon}}$.

* If we choose n large enough (larger than some suitable real number R),

then

* We can assure that x_n is as close as we want to L (we can assure that $|x_n - L|$ is within a prescribed tolerance ε).

Example 1:

Proposition

$$\lim_{n\to\infty} \frac{1}{n^2} = 0.$$

Proof. Let $\varepsilon > 0$. [Scratchwork, NOT part of the proof: we want to assure that

$$\left|\frac{1}{n^2} - 0\right| < \varepsilon.$$

This is the same as

$$\frac{1}{n^2} < \varepsilon \qquad \text{(since } n \in \mathbb{N}\text{, so } \tfrac{1}{n^2} > 0\text{).}$$

Solve for n:

$$n^2 > 1/\varepsilon$$
$$n > 1/\sqrt{\varepsilon}.$$

So, as long as $n > 1/\sqrt{\varepsilon}$, we'll have what we need. This tells us how big R needs to be. Now write this:]

Suppose $R = 1/\sqrt{\varepsilon}$.
Then

$$n > R \Rightarrow \left| \frac{1}{n^2} - 0 \right| = \frac{1}{n^2} < \frac{1}{R^2}$$
$$= \frac{1}{(1/\sqrt{\varepsilon})^2} = \frac{1}{1/\varepsilon} = \varepsilon.$$

So $\lim\limits_{n \to \infty} \frac{1}{n} = 0.$ ATWMR

The point is: a limit proof is constructive in that, for any given $\varepsilon > 0$, we prove an $R \in \mathbb{R}$ with the desired property* exists by constructing such an R.

*Namely, $n > R \Rightarrow |x_n - L| < \varepsilon$.

Limit proof template:

Proposition

$$\lim_{n\to\infty} x_n = L.$$

Proof

Let $\varepsilon > 0$. [Do some algebra to find how large n should be, to assure that $|x_n - L| < \varepsilon$. Suppose you show that $n >$ whatever works. Write this:]*

Let $R =$ whatever. Then

$n > R \Rightarrow$ (do the algebra to show that)*
$|x_n - L| < \varepsilon$.

So $\lim_{n\to\infty} x_n = L$. ATWMR.

*DON'T show this "scratchwork" in the proof.

*DO show this algebra in the proof.

$1, \frac{1}{2^2}, \frac{1}{3^2}, \cdots$

$1, \frac{1}{2^2}, 1, \frac{1}{3^2}, 1, \frac{1}{4^2}, 1, \frac{1}{5^2}, 1, \frac{1}{6^2}, \cdots$