Two last things.

A) One last induction thing.

(Compare with Exercise 12, HW #7.)

Proposition.

$\forall n \in \mathbb{N}$, $24 \mid (5^{2n} - 1)$.

Proof.

Let $A(n)$ be the statement $24 \mid (5^{2n} - 1)$.

Step 1: Is $A(1)$ true?

Does $24 \mid (5^{2\cdot 1} - 1)$?

$24 \mid 24$, so $A(1)$ is true.

Step 2: Assume

$A(k)$: $24 \mid (5^{2k} - 1)$.

To deduce

$A(k+1)$: $24 \mid (5^{2(k+1)} - 1)$,

we note that

$$\begin{aligned} 5^{2(k+1)} - 1 &= 5^{2k+2} - 1 \\ &= 5^{2k} - 1 + 5^{2k+2} - 5^{2k} \\ &= 5^{2k} - 1 + 5^{2k}(5^2 - 1) \\ &= 5^{2k} - 1 + 24 \cdot 5^{2k}. \end{aligned}$$

Now $24 \mid (5^{2k} - 1)$ by the induction hypothesis. Moreover, $24 \mid 24$, so $24 \mid 24 \cdot 5^{2k}$ by S-POP, Exercise B(i)-3(b). So $24 \mid (5^{2k} - 1 + 24 \cdot 5^{2k})$, by S-POP Exercise B(i)-3(a). So $A(k+1)$ follows.

Therefore, by induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

B) One last Fibonacci thing.

Recall: we define a sequence R_n of ratios of Fibonacci numbers by

$$R_n = \frac{F_{n+1}}{F_n} \quad (n \geq 1),$$

where F_n is the n^{th} Fibonacci number.

The sequence of R_n's begins

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \ldots$$

$$\approx 1, 2, 1.5, 1.6667, 1.6, 1.625, 1.6154, 1.6191, 1.6177, \ldots$$

In fact:

Theorem.

$$\lim_{n \to \infty} R_n = \Phi = \frac{1+\sqrt{5}}{2} \approx 1.61803.$$

Proof (sketch)

Suppose $\lim_{n \to \infty} R_n = L$.

We'll prove that $L = \Phi$.

To do so, recall that

$$F_{n+2} = F_{n+1} + F_n.$$

Divide through by F_{n+1}:

$$\frac{F_{n+2}}{F_{n+1}} = 1 + \frac{F_n}{F_{n+1}}.$$

this is $1/R_n$.

this is R_{n+1}.

So $\quad R_{n+1} = 1 + \dfrac{1}{R_n}.$ $\qquad (*)$

Now note that:

(a) $\displaystyle\lim_{n\to\infty} \frac{1}{R_n} = \frac{1}{\lim_{n\to\infty} R_n} = \frac{1}{L}$, by limit laws.

(b) As n approaches ∞, so does $n+1$, so

$$\lim_{n\to\infty} R_{n+1} = \lim_{n\to\infty} R_n = L.$$

So $(*)$, in the limit as $n\to\infty$, gives

$$L = 1 + \frac{1}{L}.$$

Multiply through by L:
$$L^2 = L+1$$

or

$$L^2 - L - 1 = 0.$$

Solve for L using the quadratic formula:

$$L = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Now all of the R_n's are positive, so their limit L can't be negative, so L can't equal $(1-\sqrt{5})/2$, so we conclude that

$$L = \frac{1+\sqrt{5}}{2} = \Phi!$$

ATWMR.