Fibonacci numbers.

Consider the sequence of natural numbers that starts with

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

These are the Fibonacci numbers F_n, defined recursively by

$$F_1 = 1,\ F_2 = 1, \qquad (*)$$

$$F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 1. \qquad (*')$$

Properties of Fibonacci numbers are often proved by induction, using the "initial values" $(*)$ and the "recursion formula" $(*')$. For example:

Proposition.
$\forall n \in \mathbb{N}$,
$$F_{n+3}F_n - F_{n+1}F_{n+2} = (-1)^{n+1}.$$

Proof.
Let $A(n)$ be the identity claimed.

Step 1: is $A(1)$ true?

$$F_4 F_1 - F_2 F_3 \stackrel{?}{=} (-1)^{1+1}$$
$$3 \cdot 1 - 1 \cdot 2 \stackrel{?}{=} (-1)^2$$
$$1 = 1 \checkmark$$

So $A(1)$ is true.

Step 2: Assume
$$A(k):\ F_{k+3}F_k - F_{k+1}F_{k+2} = (-1)^{k+1}.$$
To deduce
$$A(k+1):\ F_{k+4}F_{k+1} - F_{k+2}F_{k+3} = (-1)^{k+2},$$
we note that, by (*),

$$\underbrace{F_{k+4}}_{\text{use }(*)}F_{k+1} - \underbrace{F_{k+2}}_{\text{use }(*)}F_{k+3}$$
$$= (F_{k+3}+F_{k+2})F_{k+1} - (F_{k+1}+F_k)F_{k+3}$$
$$= \cancel{F_{k+3}F_{k+1}} + F_{k+2}F_{k+1} - \cancel{F_{k+1}F_{k+3}} - F_kF_{k+3}$$
$$= F_{k+2}F_{k+1} - F_kF_{k+3} = -(F_kF_{k+3} - F_{k+2}F_{k+1})$$
$$= -(-1)^{k+1} = (-1)^{k+2}.$$ So $A(k+1)$ follows.

Therefore, $A(k) \Rightarrow A(k+1)$.
So, by induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

COOL FACT (proof omitted: see S-POP, Proposition B(vi)-1): there's a <u>closed formula</u> for the n^{th} Fibonacci number:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \quad (n \geq 1).$$

Remark: the number $\frac{1+\sqrt{5}}{2}$ appearing above is called the <u>golden ratio</u> denoted Φ.

More on Φ and F_n soon.