More induction proofs.

I) A generalization of the product rule from calculus.

Recall that, if f_1, f_2 are differentiable functions, then

$$(f_1 f_2)' = f_1' f_2 + f_2' f_1.$$

This takes a nice form if we divide through by $f_1 f_2$:

$$\frac{(f_1 f_2)'}{f_1 f_2} = \frac{f_1' f_2}{f_1 f_2} + \frac{f_2' f_1}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}.$$

This last formula generalizes:

Theorem.
Let $n \in \mathbb{N}$; suppose $f_1, f_2, \dots, f_n$ are differentiable. Then

$$\frac{(f_1 f_2 \cdots f_n)'}{f_1 f_2 \cdots f_n} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \dots + \frac{f_n'}{f_n}.$$

Proof.
Let $A(n)$ be the statement of the theorem.
Step 1: is $A(1)$ true?

$$\frac{f_1'}{f_1} \stackrel{?}{=} \frac{f_1'}{f_1} \ \checkmark \qquad \text{So } A(1) \text{ is true.}$$

Step 2: Assume

$$A(k): \frac{(f_1 f_2 \cdots f_k)'}{f_1 f_2 \cdots f_k} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \cdots + \frac{f_k'}{f_k}.$$

Then
$$\frac{(f_1 f_2 \cdots f_{k+1})'}{f_1 f_2 f_3 \cdots f_{k+1}} = \frac{\big((f_1 f_2 \cdots f_k)\, f_{k+1}\big)'}{(f_1 f_2 \cdots f_k)\, f_{k+1}}.$$

Use the product rule in the numerator (thinking of $f_1 f_2 \cdots f_k$ as a single function) to get

$$\frac{(f_1 f_2 \cdots f_{k+1})'}{f_1 f_2 f_3 \cdots f_{k+1}} = \frac{(f_1 f_2 \cdots f_k)' f_{k+1} + (f_1 f_2 \cdots f_k) f_{k+1}'}{f_1 f_2 f_3 \cdots f_{k+1}}$$

algebra
$$= \frac{(f_1 f_2 \cdots f_k)'}{f_1 f_2 \cdots f_k} + \frac{f_{k+1}'}{f_{k+1}}$$

induction hypothesis
$$= \frac{f_1}{f_1'} + \frac{f_2'}{f_2} + \cdots + \frac{f_k'}{f_k} + \frac{f_{k+1}'}{f_{k+1}},$$

so $A(k+1)$ follows.

So by induction, $A(n)$ is true $\forall n \in \mathbb{N}$. ATWWMR.

II) A surprising fact.

Theorem.
All sneakers are identical.

Proof

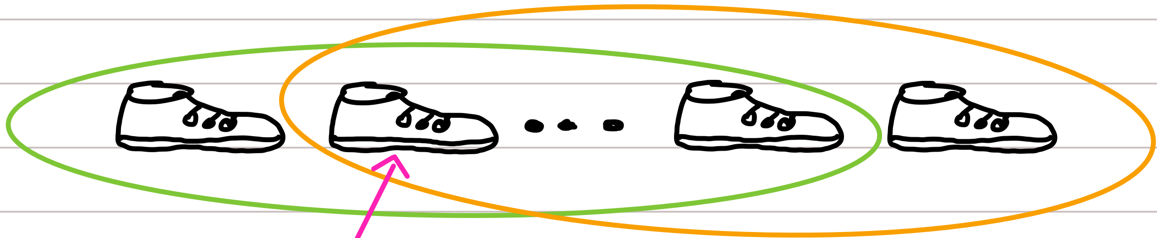
Let $A(n)$ be the statement:
Any n sneakers are identical.

We prove, by induction, that $A(n)$ is true $\forall n \in \mathbb{N}$.

Step 1: Is $A(1)$ true?
Any one sneaker is identical to itself, so yes, $A(1)$ is true.

Step 2: Assume
$A(k)$: any k sneakers are identical.

Now suppose we have $k+1$ sneakers.
Line them up:



(The second sneaker is part of the first group of k and the last)

By the induction hypothesis, the first k are identical, as are the last k.
So all $k+1$ are identical to the second one, and thus to each other.

So $A(k) \Rightarrow A(k+1)$.

So by induction, $A(n)$ is true $\forall n \in \mathbb{N}$. ATWMR

Problem with proof: The inductive step <u>fails</u> for $k=1$! (If $k=1$, the second sneaker is <u>not</u> part of both the first k and the last k.)