More on mathematical induction.

Theorem.

$\forall n \in \mathbb{N}$:

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof.

Let $A(n)$ be the above statement.

Step 1: Is $A(1)$ true?

$$1^2 \stackrel{?}{=} \frac{1(1+1)(2\cdot 1+1)}{6}$$

$$1 = 1. \checkmark$$

So $A(1)$ is true.

Step 2: Assume

$$A(k): 1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Then

$$\begin{aligned}
&1^2 + 2^2 + \dots + (k+1)^2 \\
&= (1^2 + 2^2 + \dots + k^2) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)\big(k(2k+1) + 6(k+1)\big)}{6} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{2}
\end{aligned}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)((k+1)+1)(2(k+1)+3)}{6}.$$

So $A(k+1)$ is true.

So $A(k) \Rightarrow A(k+1)$.

By induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$