Quantifiers.

The symbols $\exists$ and $\forall$ are called quantifiers.

(1) $\exists$ is the existential quantifier; it means "there exists" or "for some" or "for at least one."

Also, if $Q(x)$ is a statement about an object x, and A is a set, then
$$\exists x \in A : Q(x)$$
means $Q(x)$ holds for at least one element of A.

Examples:
$\exists x \in \mathbb{N} : x \geq 42$ is true.
$\exists x \in \mathbb{N} : x < 0$ is false.

(2) $\forall$ is the universal quantifier; it means "for all" or "for every."

Also, if $Q(x)$ is a statement regarding an object x, and A is a set, then
$$\forall x \in A : Q(x)$$
means $Q(x)$ holds for every $x \in A$.

Examples:
$\forall x \in \mathbb{N} : x \geq 42$ is false.
$\forall x \in \mathbb{N} : x \geq 0$ is true.

(3) (a) We can string quantifiers together.
Examples:

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}: x > y$ is true.
$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}: x > y$ is false.
(note that order matters!)

(b) We can negate statements with quantifiers. In particular,

$\sim (\exists x \in A : Q(x))$
is equivalent to
$\forall x \in A : \sim Q(x)$
and
$\sim (\forall x \in A : Q(x))$
is equivalent to
$\exists x \in A : \sim Q(x)$.

(4) Examples: True or False, and why?

(a) $\sim (\forall n \in \mathbb{N}, n^2 = n)$
(b) $\sim (\exists n \in \mathbb{N} \; n^2 < n)$
(c) $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}: x > y$
(d) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}: x > y$
(e) $\sim (\exists y \in \mathbb{R}, \exists x \in \mathbb{R}: |x - y| < 0)$
Also, rewrite this statement without using $\forall$.

SOLUTION.

(a) This statement is equivalent to
$\exists n \in \mathbb{N} : n^2 \neq n$
which is true (take $n = 2$).

(b) Equivalent to
$\forall n \in \mathbb{N}, n^2 \geq n$,
which is true. (Proof: if $n \in \mathbb{N}$, then $n \geq 1$.

Multiply by n to get $n^2 \geq n$.)

(c) True. Proof: given $x \in \mathbb{R}$, let $y = x - 1$.
Then $x > y$.

(d) False. There exists no real number x that's
$>$ every real number y.

(e) $\sim(\exists y \in \mathbb{R}, \exists x \in \mathbb{R} : |x-y| < 0)$
is equivalent to
$\forall y \in \mathbb{R}, \sim(\exists x \in \mathbb{R} : |x-y| < 0)$
which is equivalent to
$\forall y \in \mathbb{R}, \forall x \in \mathbb{R}, |x-y| \geq 0$,
which is true (since $|z| \geq 0$ for all real
numbers z).