

Indexed sets, union, intersection.

The idea: suppose we have a set A_α for each α in some set I . Then:

(a) We call each $\alpha \in I$ an index, or a subscript.

(b) We call I an indexing set.

(c) We define:

$$\bigcup_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for some } \alpha \in I\}$$

("the union of the A_α 's over α in I "),

$$\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for all } \alpha \in I\}$$

("the intersection of the A_α 's over α in I ").

Example 1.

$$\text{Let } \mathbb{R}^+ = \{\text{positive real numbers}\} \\ = \{x \in \mathbb{R} : x > 0\}.$$

For each $\alpha \in \mathbb{R}^+$, define

$$A_\alpha = \{x \in \mathbb{R} : -\alpha < x < \alpha\} = (-\alpha, \alpha).$$

Then

$$\bigcup_{\alpha \in \mathbb{R}^+} A_\alpha = \mathbb{R}, \quad \bigcap_{\alpha \in \mathbb{R}^+} A_\alpha = \{0\}.$$

Example 2.

$$\text{Let } P = \{\text{positive prime numbers}\} \\ = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \dots\}$$

Then

$$\bigcup_{p \in P} p\mathbb{Z} = \mathbb{Z} - \{\pm 1\}, \quad \bigcap_{p \in P} p\mathbb{Z} = \{0\}$$

* Every integer except ± 1 or -1 is divisible by some prime

* 0 is the only integer divisible by all primes (because there are infinitely many primes)

Remark. Suppose our indexing set I is a set of consecutive integers ranging from a to b (where a and/or b might be infinite). Then we write

$$\bigcup_{i=a}^b A_i \quad \text{for} \quad \bigcup_{i \in I} A_i$$

and

$$\bigcap_{i=a}^b A_i \quad \text{for} \quad \bigcap_{i \in I} A_i.$$

Example 3.

$$\bigcup_{i=0}^5 (i+6\mathbb{Z}) = \mathbb{Z} \quad \text{and} \quad \bigcap_{i=0}^5 (i+6\mathbb{Z}) = \emptyset,$$

by the division algorithm.

Example 4.

For each $i \in \mathbb{N}$, define

$$S_i = \left(\frac{1}{i+5}, \frac{1}{i} \right].$$

Then, for example,

③

$$\bigcup_{i=1}^4 S_i = \left(\frac{1}{6}, 1\right] \cup \left(\frac{1}{7}, \frac{1}{2}\right] \cup \left(\frac{1}{8}, \frac{1}{3}\right] \cup \left(\frac{1}{9}, \frac{1}{4}\right] \\ = \left(\frac{1}{9}, 1\right],$$

$$\bigcap_{i=1}^4 S_i = \left(\frac{1}{6}, \frac{1}{4}\right] \quad (\text{think about it!}),$$

$$\bigcup_{i=2}^{\infty} S_i = \left(0, \frac{1}{2}\right], \quad \bigcap_{i=2}^{\infty} S_i = \emptyset. *$$

* For example, $S_2 \cap S_8 = \left(\frac{1}{7}, 1\right] \cap \left(\frac{1}{14}, \frac{1}{8}\right] = \emptyset.$

Example 5.

For each $r \in [0, 1]$, define

$$C_r = [r, 1] \times [0, r].$$

Then

$$\bigcup_{r \in [0, 1]} C_r = \{(x, y) \in [0, 1] \times [0, 1] : y \leq x\}$$

and

$$\bigcap_{r \in [0, 1]} C_r = \{(1, 0)\}.$$

Picture:



