# More proof strategies/templates.

Notation: if $P$ is any statement, then $\sim P$ denotes the negation, called "not $P$" of $P$. (Sometimes, $\sim P$ is written $\neg P$.)

## 1) Contrapositive.

**Fact.** $P \Rightarrow Q$ is logically equivalent to $\sim Q \Rightarrow \sim P$. (These statements are true or false together.)

Think about it: if $P$ necessitates $Q$, then for $Q$ to fail, $P$ must fail too.

Here's a proof by contrapositive (or by contraposition) template:

**Theorem.** $P \Rightarrow Q$.

**Proof.**
Assume $\sim Q$.
[Now do stuff to get:] Therefore, $\sim P$.

So $P \Rightarrow Q$. □

**Example.**
Let $m \in \mathbb{Z}$. Prove that, if $m^2$ is odd, then $m$ is odd.

**Solution.**
Let $m \in \mathbb{Z}$. Suppose $m$ is not odd.

Then $m$ is even. So $m = 2k$ for some $k \in \mathbb{Z}$. But then

$$\begin{aligned} m^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \\ &= 2n, \end{aligned}$$

where $n = 2k^2 \in \mathbb{Z}$. So $m^2$ is even. So $m^2$ is not odd.

So if $m^2$ is odd, then $m$ is odd. □

2) $P \Leftrightarrow Q$.

By definition, $P \Leftrightarrow Q$ (also written $P$ iff $Q$, and read "$P$ if and only if $Q$," or "$P$ iffy $Q$") means $P \Rightarrow Q$ and $Q \Rightarrow P$.

$P \Leftrightarrow Q$ proof template:

Theorem. $P \Leftrightarrow Q$.

Proof.

1) Assume $P$. [Then do stuff to show:] Therefore, $Q$.
2) Next, assume $Q$. [Then do stuff to show:]: Therefore, $P$.

So $P \Leftrightarrow Q$. □

[Remark: alternatively, in a $P \Leftrightarrow Q$ proof, you can prove $P \Rightarrow Q$ or* $Q \Rightarrow P$ by contraposition.]

* Inclusive or.

Example:

**Theorem.** Let $n \in \mathbb{Z}$. Then $n$ is even iff $n^2+6n+5$ is odd.

**Proof.**

Let $n \in \mathbb{Z}$.

1) Assume $n$ is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. So

$$\begin{aligned} n^2+5n+6 &= (n+1)(n+5) \\ &= (2k+1)(2k+5) \\ &= (2k+1)(2(k+2)+1) \\ &= (2k+1)(2m+1), \end{aligned}$$

where $k, m \in \mathbb{Z}$. So $n^2+5n+6$ is a product of two odd numbers, and is therefore odd, by Exercise B(i)-1b in S-POP.

2) Next, assume $n$ is <u>not</u> even. Then $n$ is odd, so $n = 2k+1$ for some $k \in \mathbb{Z}$. So

$$\begin{aligned} n^2+6n+5 &= (n+1)(n+5) \\ &= (2k+1+1)(2k+1+5) \\ &= (2k+2)(2k+6) \\ &= (2(k+1))(2(k+3)) \\ &= (2m)(2l), \end{aligned}$$

where $m, l \in \mathbb{Z}$. So $n^2+6n+5$ is a product of even numbers, and is therefore even and is not odd, by Exercise B(i)-1a in S-POP.

So $n$ is even iff $n^2+6n+5$ is odd.

□