Intro to proofs: the statement $P \Rightarrow Q$.

Let P, Q be any statements, like: "it's raining," "n is a perfect square," "$A \subseteq B$," "39 is even."

The following all mean the same thing:

- P implies Q,
- $P \Rightarrow Q$,
- If P then Q,
- Whenever P is true, Q follows.
- Under the condition P, the conclusion Q holds.

Example: if P is "it's Friday" and Q is "it's the weekend," then $P \Rightarrow Q$. However, $Q \not\Rightarrow P$.

To prove a $P \Rightarrow Q$ statement, assume P, then do whatever works to conclude Q.

$P \Rightarrow Q$ proof TEMPLATE:

Theorem. $P \Rightarrow Q$.

Proof. Assume P. [Then do what you have to, to get to:] Therefore, Q.

So $P \Rightarrow Q$. $\square$

Notes.

1) The last line, "So $P \Rightarrow Q$," is optional.

2) The "$\square$" is to clearly indicate end of proof.

EXAMPLES.

Theorem 1.
If n is an even integer, then $n-1$ is an odd integer.
Proof.
Assume n is an even integer. Then $n = 2k$ for some $k \in \mathbb{Z}$. Therefore, $n-1 = 2k-1$ for some $k \in \mathbb{Z}$. So $n-1$ is odd.

$\underbrace{\text{Therefore, } n \text{ is even} \Rightarrow n-1 \text{ is odd.}}_{\text{optional statement}}$ $\square$

[Note: if $a, b \in \mathbb{Z}$, we say "a divides b," written $a|b$, if $b = an$ for some $n \in \mathbb{Z}$.
E.g. $3|6$ (since $6 = 3 \cdot 2$); $7|0$ (since $0 = 7 \cdot 0$); $7 \nmid 15$.]

Proposition 1.*
Let $b \in \mathbb{Z}$. If $10|b$, then $5|b$.
Proof.
Assume $b \in \mathbb{Z}$ and $10|b$. Then $b = 10n$ for some $n \in \mathbb{Z}$. But $10 = 5 \cdot 2$, so
$b = (5 \cdot 2)n = 5 \cdot 2n$. So $5|b$.

So $10|b \Rightarrow 5|b$. $\square$

* A proposition is like a theorem, only less significant. (Significance is entirely subjective.)

[Question: is the converse to Proposition 1 true? That is: does $5 | b \Rightarrow 10 | b$, for $b \in \mathbb{Z}$?

Answer: no. For example, $5 | 25$, but $10 \nmid 25$.]

Theorem 2.

If $n \in \mathbb{Z}$, then $n^2 + 5n + 4$ is even.

Proof.

Assume $n \in \mathbb{Z}$. We consider two cases: **

(a) Suppose n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. So

$$\begin{aligned} n^2 + 5n + 4 &= (2k)^2 + 5(2k) + 4 \\ &= 4k^2 + 10k + 4 \\ &= 2(2k^2 + 5k + 2) \\ &= 2m, \end{aligned}$$

where $m = 2k^2 + 5k + 2 \in \mathbb{Z}$. So $n^2 + 5n + 4$ is even.

(b) Suppose n is odd. Then $n = 2k+1$ for some $k \in \mathbb{Z}$. So

$$\begin{aligned} n^2 + 5n + 4 &= (2k+1)^2 + 5(2k+1) + 4 \\ &= 4k^2 + 4k + 1 + 10k + 5 + 4 \\ &= 4k^2 + 14k + 10 \\ &= 2(2k^2 + 7k + 5) \\ &= 2\ell, \end{aligned}$$

where $\ell = 2k^2 + 7k + 5 \in \mathbb{Z}$. So $n^2 + 5n + 4$ is even.

Now n must be even or odd, and in either case, $n^2 + 5n + 4$ is even.

So $n \in \mathbb{Z} \Rightarrow n^2 + 5n + 4$ is even. $\square$

** This method is called <u>proof by cases.</u>