

Monday, 2/5 - ①

More on proofs.

Part I: $A=B$.

Let A and B be sets. To say $A=B$ is to say

$$A \subseteq B$$

and

$$B \subseteq A$$

which is to say

$$x \in A \Rightarrow x \in B$$

and

$$x \in B \Rightarrow x \in A$$

which is to say

$$x \in A \Leftrightarrow x \in B.$$

So an $A=B$ proof is a kind of $P \Leftrightarrow Q$ proof.

Template:

Theorem.

$$A=B.$$

Proof.

1) Let $x \in A$. [Do stuff to get to:] therefore, $x \in B$. So $A \subseteq B$.

2) Next, let $x \in B$. [Do stuff to get to:] Therefore, $x \in A$. So $B \subseteq A$.

$$\text{So } B=A.$$

□

Example:

Theorem.

For any sets X , Y , and Z ,

$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z).$$

Proof.

Let X, Y , and Z be sets.

- 1) Assume $x \in X \cap (Y \cup Z)$. Then, by definition of union, $x \in X$ and $x \in Y \cup Z$. So by definition of union, $x \in Y$ or $x \in Z$.

We consider two cases:

(a) $x \in X$ and $x \in Y$. Then, by definition of intersection, $x \in X \cap Y$. But then, by definition of union, $x \in (X \cap Y) \cup (X \cap Z)$.

(b) $x \in X$ and $x \in Z$. Then, by definition of intersection, $x \in X \cap Z$. But then, by definition of union, $x \in (X \cap Y) \cup (X \cap Z)$.

In either case, $x \in (X \cap Y) \cup (X \cap Z)$. So $X \cap (Y \cup Z) \subseteq (X \cap Y) \cup (X \cap Z)$.

- 2) Assume $x \in (X \cap Y) \cup (X \cap Z)$. Then, by definition of union, $x \in X \cap Y$ or $X \cap Z$. In either case, $x \in X$. Moreover, in the first case $x \in Y$, by definition, while in the second case, $x \in Z$, by the same definition. So in either case, $x \in Y \cup Z$, by the definition of union. So in both cases, $x \in X \cap (Y \cup Z)$, by definition of intersection. So $(X \cap Y) \cup (X \cap Z) \subseteq X \cap (Y \cup Z)$.

Therefore, $(X \cap Y) \cup (X \cap Z) = X \cap (Y \cup Z)$.

□

Part II: Disproof; counterexamples.

To say a statement P is true is to say it's true no matter what. So to disprove P , it's enough to give a counterexample.

Examples:

1) Are all prime numbers odd?

No: 2 is prime but not odd.

2) Is the following true?

Proposition.

For all sets A, B , and C ,

$$A - (B \cap C) = (A - B) \cap (A - C).$$

Solution:

This is false. Counterexample:

$$A = \mathbb{Z}, \quad B = 1 + 3\mathbb{Z}; \quad C = 2 + 3\mathbb{Z}.$$

Then $B \cap C = \emptyset$. (See p.4 of S-POB, or Fact 1.5, p. 30 of T-BOB, on the division algorithm.)

$$\text{So } A - (B \cap C) = A = \mathbb{Z}.$$

But

$$A - B = 3\mathbb{Z} \cup 2 + 3\mathbb{Z} \quad \text{and}$$

$$A - C = 3\mathbb{Z} \cup 1 + 3\mathbb{Z}, \quad \text{so}$$

$$(A - B) \cap (A - C) = 3\mathbb{Z} \neq A - (B \cap C).$$