HW3 Math 2001 Bodhi Rubinstein

S-POP Section B(i) - 4, 5, 6, 7, 8, 9, 10

$\boxed{4.}$ Theorem: If n is an even number then $n-1$ is an odd number

Proof: Assume $n-1$ is not an odd number.

Then $n-1$ is an even number.

Then $n-1 = 2k$ for some $k \in \mathbb{Z}$.

But then, $n = 2k+1$, so by definition n is odd.

So n is not even.

So if n is even, the $n-1$ is odd. $\square$

$\boxed{5.}$ Theorem: If m^2 is an odd number, then m is an odd number

Proof: Assume m^2 is odd.

Then, by definition $m^2 = 2l+1$ for some $l \in \mathbb{Z}$.

Since m is either even or odd, it follows that $m = 2k+r$ for some $k \in \mathbb{Z}$ and r equal to 0 or 1.

It follows that $m^2 = (2k+r)^2 = 4k^2 + 4kr + r^2$.

Case 1: If $r=1$, then $m = 2k+1$ for some $k \in \mathbb{Z}$ and m is odd.

So, $m^2 = 4k^2+4k+1 = 2(2k^2+2k)+1 = 2p+1$, where $p = 2k^2+2k$, $p \in \mathbb{Z}$. Therefore m^2, by definition, is odd.

Case 2: If $r=0$, then $m = 2k$ for some $k \in \mathbb{Z}$ and m is even

So, $m^2 = 4k^2 = 2(2k^2) = 2v$, where $v = 2k^2$, $v \in \mathbb{Z}$.

Therefore, m^2 by definition is even.

But, m^2 is not even so r must equal 1 and m must be odd

So if m^2 is an odd number, then m is an odd number. $\square$

$\boxed{6.}$ Theorem: If $4 \nmid n$, then $12 \nmid n$

Proof: Assume $12 | n$.

Then $n = 12b$, where $b \in \mathbb{Z}$.

Then since $12 = 3 \cdot 4$, $n = (3 \cdot 4)b = (4) \cdot (3b) = 4l$, where $l = 3b$, $l \in \mathbb{Z}$.

Then since $n = 4l$, by definition $4 | n$.

So if $4 \nmid n$, then $12 \nmid n$. $\square$

$\boxed{7.}$ Theorem: The statement that if x is odd, then x is divisible by 3 is false.

Proof: Assume $x = 5$.

Then $x = 2(2) + 1 = 2k + 1$, where $k = 2$.

Therefore x is odd.

By definition, if $3 | 5$, then $5 = 3n$ for some $n \in \mathbb{Z}$.

But $5 \neq 3n$ for any $n \in \mathbb{Z}$.

Therefore $3 \nmid 5$ and $x = 5$ is odd.

So the statement that if x is odd, then x is divisible by 3 is false. $\square$

$\boxed{8.}$ Theorem: the converse statement of 3(i)-3(a), that if $a | (b+c)$, then $a | b$ and $a | c$, is false.

Proof: Let $a = 3, b = 2, c = 1$.

Then $a | (b+c) = 3 | (2+1) = 3 | 3$.

By definition, $3 | 3$ because $3 = 3n$ where $n = 1$.

Then $a | b = 3 | 1$, but $3 \nmid 1$ because $1 \neq 3k$ for any $k \in \mathbb{Z}$.

Similarly, $a | c = 3 | 2$, but $3 \nmid 2$ because $2 \neq 3p$ for any $p \in \mathbb{Z}$.

So the statement that if $a | (b+c)$, then $a | b$ and $a | c$, is false.

$\square$

9.) **Theorem:** For $n \in \mathbb{Z}$, $6|n$ if and only if n is even and $3|n$.

Proof:

a) Assume $n \in \mathbb{Z}$ and $6|n$.
Then $n = 6k$ for some $k \in \mathbb{Z}$.
Then $n = (2 \cdot 3)k = 3(2k) = 2(3k) = 3l = 2p$, where $l = 2k, l \in \mathbb{Z}$ and $p = 3k, p \in \mathbb{Z}$.
Therefore $3|n$ and $2|n$ and n is even.
So for $n \in \mathbb{Z}$, if $6|n$ the n is even and $3|n$.

b) Assume $n \in \mathbb{Z}$, n is even, and $3|n$
Then, $n = 2k$ for some $k \in \mathbb{Z}$ and $n = 3m$ for some $m \in \mathbb{Z}$.
Since, by Exercise B(i)-1(b) the product of two odd numbers is odd, if m is odd then $m \cdot 3$ is odd.
But since $n = 2k$, $2k = 3m$ then $3 \cdot m$ must be even.
Therefore m must be even and $m = 2l$ for some $l \in \mathbb{Z}$.
So, $n = 3m = 3(2l) = 6l$.
Then by definition $6|n$.
So for $n \in \mathbb{Z}$, if n is even and $3|n$, then $6|n$.
Therefore, for $n \in \mathbb{Z}$, $6|n$ iff n is even and $3|n$. □

10. **Theorem:** $x^2 = 1$ iff $x = -1$ or $x = 1$

Proof: Assume $x^2 = 1$.
Then $x^2 - 1 = 0 = (x+1)(x-1) = ab$ for $a = x+1$, $b = x-1$.
Since $a \cdot b = 0$ iff $a = 0$ or $b = 0$, then $x^2 - 1 = 0$ iff $x + 1 = 0$ or $x - 1 = 0$.
Therefore, $x = -1$ or $x = 1$.
So, $x^2 = 1$ iff $x = -1$ or $x = 1$.

S-POP Section B(ii): Exercises 5,6,7

5. Theorem: $\mathbb{Z} = 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$

Proof:

a) Let $x \in \mathbb{Z}$.

By the division algorithm, $x = 3q + r$ for some $q \in \mathbb{Z}$ and $r = 0$ or 1 or 2.

If $r=0$, then $x=3q$, so $x \in 3\mathbb{Z}$

If $r=1$, then $x=3q+1$, so $x \in 3\mathbb{Z}+1$

If $r=2$, then $x=3q+2$, so $x \in 3\mathbb{Z}+2$

Therefore $x \in 3\mathbb{Z}$ or $x \in 3\mathbb{Z}+1$ or $x \in 3\mathbb{Z}+2$.

So, by definition, $\mathbb{Z} \subseteq 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$.

b) Let $x \in 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$.

Then, by def., $x \in 3\mathbb{Z}$ or $x \in 3\mathbb{Z}+1$ or $x \in 3\mathbb{Z}+2$.

If $x \in 3\mathbb{Z}$, then $x = 3k$ for some $k \in \mathbb{Z}$, so $x \in \mathbb{Z}$

If $x \in 3\mathbb{Z}+1$, then $x = 3l+1$ for some $l \in \mathbb{Z}$, so $x \in \mathbb{Z}$

If $x \in 3\mathbb{Z}+2$, then $x = 3p+2$ for some $p \in \mathbb{Z}$, so $x \in \mathbb{Z}$

Therefore, $x \in \mathbb{Z}$ and $x \in 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$.

So, $3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2 \subseteq \mathbb{Z}$.

So, $\mathbb{Z} = 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$. □

6. Theorem: $3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$ is disjoint

Proof: Let $x \in 3\mathbb{Z}$, $y \in 3\mathbb{Z}+1$, $z \in 3\mathbb{Z}+2$

By the division algorithm, $x = 3q$, $y = 3q+1$, $z = 3q+2$ for some $q \in \mathbb{Z}$.

Since $3 | 3q$ with no remainder, $3 | 3q+1$ with a remainder of 1, and $3 | 3q+2$ with a remainder of 2, the x, y, z are all unique integers.

Therefore $3\mathbb{Z}, 3\mathbb{Z}+1, 3\mathbb{Z}+2$ have no common elements

So $3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$ is disjoint. □

7. Theorem: For any sets X, Y, Z, $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.
Proof: a) Let $x \in X \cup (Y \cap Z)$.
Then, by def., $x \in X$ or $x \in Y \cap Z$.
If $x \in X$, then $x \in X$ or $x \in Y$ or $x \in Z$.
So, $x \in X \cup Y$ and $x \in X \cup Z$.
So, by def., $x \in (X \cup Y) \cap (X \cup Z)$.
If $x \in Y \cap Z$, then $x \in Y$ and $x \in Z$
Therefore, $x \in X$ or $x \in Y$, so by def., $x \in X \cup Y$.
Similarly, $x \in X$ or $x \in Z$, so by def., $x \in X \cup Z$.
So, by def., $x \in (X \cup Y) \cap (X \cup Z)$
Therefor $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$
b) Let $x \in (X \cup Y) \cap (X \cup Z)$.
Then, $x \in (X \cup Y)$ and $x \in X \cup Z$.
If $x \in X \cup Y$, then $x \in X$ or $x \in Y$
If $x \in X \cup Z$, then $x \in X$ or $x \in Z$
So, since $x \in X$ in both case, by def. $x \in X \cup (Y \cap Z)$.
So, $(X \cup Y) \cap (X \cup Z) \subseteq X \cup (Y \cap Z)$
Therefore, for any sets X, Y, Z, $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$. $\square$

T-BOP Chapter 8: Exercises 14, 16
14. Theorem: For any sets A, B, C, $(A \cup B) - C = (A - C) \cup (B - C)$
Proof: a) Let $x \in (A \cup B) - C$. Then $x \in A$ or $x \in B$, but $x \notin C$.
If $x \in A$, then $x \notin C$, so $x \in A - C$
If $x \in B$, then $x \notin C$, so $x \in B - C$
So, by def., $x \in (A - C) \cup (B - C)$
So $(A \cup B) - C \subseteq (A - C) \cup (B - C)$
b) Let $x \in (A - C) \cup (B - C)$. then $x \in A - C$ or $x \in B - C$.
If $x \in A - C$, then $x \in A$ but $x \notin C$. Therefore $x \in A$ or $x \in B$,
so $x \in A \cup B$. Since $x \notin C$, then $x \in (A \cup B) - C$

If $x \in B - C$, then $x \in B$ but $x \notin C$. Therefore $x \in B$ or $x \in A$, so $x \in (A \cup B)$. Since $x \notin C$, then $x \in (A \cup B) - C$.
Therefore $(A - C) \cup (B - C) \subseteq (A \cup B) - C$
So, $(A \cup B) - C = (A - C) \cup (B - C)$. $\square$

16) Theorem: $A \times (B \cup C) = (A \times B) \cup (A \times C)$ for any sets A, B, C
Proof:
a) Let $(a,b) \in A \times (B \cup C)$. Then $a \in A$ and $b \in B \cup C$, so $b \in B$ or $b \in C$
If $b \in B$, then $(a,b) \in A \times B$, since $a \in A$ and $b \in B$
if $b \in C$, then $(a,b) \in A \times C$, since $a \in A$ and $b \in C$
Therefore, by def, $(a,b) \in (A \times B) \cup (A \times C)$
So, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$
b) Let $(a,b) \in (A \times B) \cup (A \times C)$, then $(a,b) \in A \times B$ or $(a,b) \in A \times C$.
If $(a,b) \in A \times B$, then $a \in A$ and $b \in B$. Since $b \in B$, then $b \in B$ or $b \in C$, so $b \in B \cup C$. Therefore $(a,b) \in A \times (B \cup C)$
If $(a,b) \in A \times C$, then $a \in A$ and $b \in C$. Since $b \in C$, then $b \in C$ or $b \in B$, so $b \in B \cup C$. Therefore, $(a,b) \in A \times (B \cup C)$
So, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$
Therefore, $A \times (B \cup C) = (A \times B) \cup (A \times C)$. $\square$