HW 3

S-BoP, section B(i):

B(i)-4:

<u>Proposition:</u> If n is an even number then $n-1$ is an odd number.

<u>Proof:</u>

Assume $n-1$ is not odd. So $n-1$ is even.

By def'n, if $n-1$ is even then $n-1 = 2k$ for some $k \in \mathbb{Z}$.

So, $n-1 = 2k$ can be re-written as $n = 2k+1 = 2(k+1)-1$.

$= 2m-1$, where $m = (k+1) \in \mathbb{Z}$.

So, $n = 2m-1$

We see that n is an odd number by def'n.

Therefore, if $n-1$ is not an odd number, then n is not an even number. □

B(i)-5:

<u>Proposition:</u> If m^2 is an odd number, then m is an odd number.

<u>Proof:</u>

Assume m^2 is odd.

So $m^2 = 2l+1$, for some $l \in \mathbb{Z}$.

Assuming that m is odd we can write $m = 2k+r$ for some $k \in \mathbb{Z}$ and r is either 0 or 1.

So we can re-write $m^2 = 2l+1$ as $(2k+r)^2 = 2l+1$. Rearranging we get $4k^2 + 4kr + r^2 = 2l+1$.

So $4k^2 + 4kr + (r^2-1) = 2l$

Since the left side of the equation is even then the right must also be even.

But $2l$ is even so r^2-1 must also be even as well.

We consider two cases:

1.) If $r=0$, then $r^2-1 = 0^2-1 = -1$, which is not an even number.

2.) If $r=1$, then $r^2-1 = 1^2-1 = 0$, which is even.

Therefore we can conclude that $r=1$. So $m = 2k+1$. So if m^2 is odd then m is odd. □

HW 3 cont.

B(i)-6:

Theorem: If n is not divisible by 4, then n is not divisible by 12.

Proof:

Assume n is divisible by 12.

Then by def'n, $n = 12x$ for some $x \in \mathbb{Z}$.

Let n by divisible by 4.

Then by def'n $n = 4k$ for some $k \in \mathbb{Z}$.

So, $12x = 4k$, which can be re-written as $3x = k$.

So, $n = 4k = 4(3x)$

so $n = 12x$.

Therefore, if n is divisible by 12 then n is divisible by 4.

So, if n is not divisible by 4, then n is not divisible by 12. $\square$

B(i)-7:

If x is odd, then x is divisible by 3.

False: $x = 5$ is odd by is not divisible by 3 as it leaves a remainder of 2.

B(i)-8:

If $a|(b+c)$, then $a|b$ and $a|c$.

False: Let $a=5$, $b=21$, and $c=4$. $a|(b+c)$ because $25 = 5k$ for some $k \in \mathbb{Z}$. If $k=5$ then $25=25$. However $b+c = 25$ but $b = 21$ and $21 \neq 5k$ and $c = 4$ and $4 \neq 5k$. where $k \in \mathbb{Z}$.

B(i)-9:

Theorem: An integer n is divisible by 6 iffy n is both even and divisible by 3.

Proof:

1: $\Rightarrow$) Assume n is divisible by 6.

Then $n = 6m$ for some $m \in \mathbb{Z}$.

By the contraposition of exercise B(i) - 1(c)

If n is even then m is also be even, so $m = 2k$ for some $k \in \mathbb{Z}$.

So $n = 6m = 6(2k) = 12k$.

$\rightarrow$

HWk 3 cont.

B(i)-9 (continued):

$n=12k$ can also be expressed as $n=2l$, where $l=6k$ and $l \in \mathbb{Z}$.
By def'n $n=2l$ is even.
n is also divisible by 3 because 12 is divisible by 3.
$\Leftarrow$:) Assume n is both even and divisible by 3.
Then if n is divisible by 3, $n=3k$ for some $k \in \mathbb{Z}$.
However, if n is also an even number then k must be divisible by 2 meaning k must be even.
So $k=2w$ for some $w \in \mathbb{Z}$.
So $n=3k$ can be re-written as $n=3(2w)=6w$, which by def'n is even.
So if $n=6w$ then n is also divisible by 6.
So n is divisible by 6 iff n is both even and divisible by 3. $\square$

B(i)-10:
Theorem: $x^2=1$ iff $x=-1$ or $x=1$.
proof:
Assume x is a real number.
$x^2=1$ iff $x^2-1=0$.
We can factor $x^2-1=(x-1)(x+1)$.
$x^2-1=0$ iff $(x-1)(x+1)=0$ iff $x-1=0$ or $x+1=0$.
Therefore, $x=1$ or $x=-1$.
So $x^2=1$ iff $x=-1$ or $x=1$. $\square$

$\rightarrow$

HWk 3 cont.

B(ii)-5:

Theorem: $\mathbb{Z} = 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$

proof:

Let $x \in \mathbb{Z}$, by the division algorithm, we can express x as $x = 3q+r$ for some $q, r \in \mathbb{Z}$ and $0 \leq r < 3$.

If $r=0$, then $x = 3q$, which means $x \in 3\mathbb{Z}$.

If $r=1$, then $x = 3q+1$, which means $x \in (3\mathbb{Z}+1)$.

If $r=2$, then $x = 3q+2$, which means $x \in (3\mathbb{Z}+2)$.

In each case $x \in 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$.

So $\mathbb{Z} \subseteq 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$.

Let $x \in 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$.

x can be expressed as $3q+r$ for some $q, r \in \mathbb{Z}$ and such that $0 \leq r < 3$.

Since $q \in \mathbb{Z} \Rightarrow 3q \in \mathbb{Z}$.

Therefore, $x \in \mathbb{Z}$.

So $3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2) \subseteq \mathbb{Z}$.

Therefore, $\mathbb{Z} \subseteq 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$ and $3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2) \subseteq \mathbb{Z}$.

So $\mathbb{Z} = 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$. $\square$

B(ii)-6:

No the sets are not disjoint.

B(ii)-7:

Theorem: $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

proof:

Let X, Y, and Z be sets.

1) Assume $k \in X \cup (Y \cap Z)$.

Then, by def'n of union $k \in X$ or $k \in (Y \cap Z)$.

Then, by def'n of intersection $k \in Y$ and $k \in Z$.

We consider two cases:

HWK 3 cont.

B(ii)-7 (continued):

(a) $k \in y$. Then, since $k \in x$ whe have $k \in (x \cup y)$, by def'n of union.
So $k \in (x \cup y) \cap (x \cup z)$ by def'n of intersection.
(b) $k \in z$. Then, since $k \in x$ we have $k \in (x \cup z)$, by def'n of union.
Therefore, $x \cup (y \cap z) \subseteq (x \cup y) \cap (x \cup z)$.
2) Assume that $k \in (x \cup y) \cap (x \cup z)$.
Then $k \in (x \cup y)$ and $k \in (x \cup z)$ by def'n of intersection.
In either case, $k \in x$ by def'n of union.
Moreover, in the first case $k \in y$ by def'n of union while in the second case $k \in z$ by def'n of union.
So in either case, $k \in y \cap z$ by def'n of intersection.
So in both cases $k \in x \cup (y \cap z)$ by def'n of union
So $(x \cup y) \cap (x \cup z) \subseteq x \cup (y \cap z)$.
Therefore, $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$. □

T-BoP chapter 8:

(14) Theorem: If A, B, and C are sets, then $(A \cup B) - C = (A - C) \cup (B - C)$.
proof:
Let A, B, and C be sets.
Assume $x \in (A \cup B) - C$.
So $x \in A \cup B$ by def'n of union and $x \notin C$ by def'n of minus.
Since $x \in A \cup B$, either $x \in A$ or $x \in B$
If $x \in A$, then $x \in A - C$ by def'n of minus.
If $x \in B$, then $x \in B - C$ by def'n of minus.
In either case, $x \in (A - C) \cup (B - C)$.
So $(A \cup B) - C \subseteq (A - C) \cup (B - C)$.
Next, assume $x \in (A - C) \cup (B - C)$.
So $x \in A - C$ or $x \in B - C$ by def'n of union.

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Hwk3 cont.

14 continued:

If $x \in A-C$, then $x \in A$ and $x \notin C$ by def'n of minus.
If $x \in B-C$, then $x \in B$ and $x \notin C$ by def'n of minus.
since $x \in A$ and $x \in B$ then $x \in A \cup B$ by def'n of union.
since $x \notin C$ then $x \notin A \cup B - C$.
so $(A-C) \cup (B-C) \subseteq (A \cup B) - C$.
since $(A \cup B) - C \subseteq (A-C) \cup (B-C)$ and $(A-C) \cup (B-C) \subseteq (A \cup B) - C$
then $(A \cup B) - C = (A-C) \cup (B-C)$. □

16 Theorem: If A, B, and C are sets, then $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
Proof:
let A, B, and C be sets.
Assume $(a,b) \in A \times (B \cup C)$.
By def'n of cartesian product $a \in A$ and $b \in B \cup C$.
By def'n of union, $b \in B$ or $b \in C$.
since, $a \in A$ and $b \in B$ it follows that $(a,b) \in A \times B$ by def'n of cartesian product.
since $a \in A$ and $b \in C$ it follows that $(a,b) \in A \times C$ by def'n of cartesian product.
so $(a,b) \in A \times B$ and $(a,b) \in A \times C$, so $(a,b) \in (A \times B) \cup (A \times C)$ by def'n of union.
Therefore, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.
Next, suppose $(a,b) \in (A \times B) \cup (A \times C)$.
By def'n of union, $(a,b) \in A \times B$ and $(a,b) \in A \times C$.
By def'n of cartesian product $(a,b) \in A \times B$ so $a \in A$ and $b \in B$.
By def'n of cartesian product $(a,b) \in A \times C$ so $a \in A$ and $b \in C$.
since, $b \in B$ and $b \in C$ than $b \in (B \cup C)$ by def'n of union.
So $a \in A$ and $b \in B \cup C$ so $(a,b) \in A \times (B \cup C)$.
Therefore, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.
since $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ and $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$
then $A \times (B \cup C) = (A \times B) \cup (A \times C)$. □