Readings

after B(i)-3

SPOP - B(i), B(ii)

TBOP - read | skim chap. 5 & 8

Excersises

SPOP - B(i) - 4 thru B(i) - 10

B(ii) - 5 thru 7

TBOP - chap 8 (171) ex. 14, 16

B(i).

4) supply a proof of contraposition for B(i) - 1_E

(note: assume $\sim Q$ . . . . therefore $\sim P$
so, $P \Rightarrow Q$)

Proposition: If n is even, then $n-1$ is odd

Proof: assume n is some number $n \in \mathbb{Z}$, and assume n is not even. Therefore n is odd and can be denoted by $n = 2k+1$ for some $k \in \mathbb{Z}$.

this means $n-1 = 2k+1-1$

$n-1 = 2k$.

so, $n-1$ is even.

Therefore by contradiction, if n is even, then $n-1$ is odd.

5) supply a direct proof of proposition. for B(i) - 2_E

Proposition: if m^2 is odd, then m is odd.

Proof: Suppose m^2 is odd so $m^2 = 2k+1$ for some $k \in \mathbb{Z}$. Then say $m = 2\ell + n$ for some $\ell \in \mathbb{Z}$ and $n \in \{0, 1\}$, meaning m will be either even $(n=0)$ or odd $(n=1)$.

$$m^2 = m^2$$
$$2k+1 = (2\ell+n)(2\ell+n)$$
$$2k+1 = 4\ell^2 + 2\ell n + 2\ell n + n^2$$
$$2k+1 = 4\ell^2 + 4\ell n + n^2$$

Proof by cases!

1) where $n=0$,

⟶ next pg.

$2k+1 = 4l^2 + 4ln + n^2$ ★ $a|b \Rightarrow$ a divisible by b if exists some $q \exists \mathbb{Z}$ such that $b = aq$

Say $n=0$,

$2k+1 = 4l^2 + 0 + 0$

$2k+1 = 2(2l^2)$

this is impossible because an even number $\neq$ an odd numbes.

2) where $n=1$

$2k+1 = 4l^2 + 4l + 1$

$2k+1 = 2(2l^2 + 2l) + 1$

we can say $b = 2l^2 + 2l$ for some number 4.

$2k+1 = 2b+1$

therefore, because the case only works if $n=1$, making m odd, we can conclude that if m^2 is odd, m must be odd.

6) using contraposition prove that, if n is not divisible by 4, then n is not divisible by 12.

<u>Proof</u>: Assume n is divisible by 4.

by definition of divisibility, $n|4$ if there exists some q such that $n=4q$. where $q \in \mathbb{Z}$. For a number n to be divisible by 12, there must be some x such that $n = 12x$. we can rewrite this to say, $n = 4(3x)$. So, by definition of divisibility it is necessary for a number to be divisible by 4 to be divisible by 12.

Therefore, if n is not divisible by 4, then n is not divisible by 12.

7) Consider:

if x is odd, then x is divisible by 3,

use counter example to prove this is wrong

say $x = 41$, which is odd because

$x = 2(20) + 1$.

but 41 is not divisible by 3 because $41 \% 3 = 2$

Therefore, by counterexample, the proof does not hold.

8) Proove (if applicable) that the converse to B(i) - 3(a) if true, else, why is it false (counterexample).

B(i) - 3a Proove that if $a \mid b$ and $a \mid c$, then $a \mid b + c$

conversely, if $a \nmid b$ and $a \nmid c$, then $a \nmid (b+c)$.

if $a \nmid b$, then $a \neq bx$ for some number x. and, if $a \nmid c$, then $a \neq cd$ for some number d. Similarly, if $a \nmid (b+c)$, then $a \neq (b+c)z$ for some number z.

$$a \neq zb + zc$$

note: $a \neq bx$

$b \neq \frac{a}{x}$

and

$a \neq cd$

$c \neq \frac{a}{d}$

these are the situations that would prove the counter, but because $a \neq bx$ and $a \neq cd$, $b \neq \frac{a}{x}$ and $c \neq \frac{a}{d}$ therefore the theorem does not hold.

counter:

say $a = 7$, $b = 4$, $c = 3$. it is true that $7 \nmid 4$ and $7 \nmid 3$.

$a \mid b + c$

$7 \mid 3 + 4$

$7 \mid 7$

therefore by definition of divisibility, if $a \nmid b$ and $a \nmid c$, it does not hold that $a \nmid b + c$.

9) <u>Theorem</u>: an integer $\underbrace{n|6}_{P}$ iff $\underbrace{n \text{ is even and } n|3}_{Q}$

<u>Proof</u>:

1) first we show that $n|6$ if n is even and $n|3$

we write that $n|6$ implies $n = 6q$ for some $n, q \in \mathbb{Z}$. This can be rewritten as $n = 2 \cdot 3 \cdot q$ which implies that n must be even and divisible by 3 if $n|6$.

2) now we show that if n is even and $n|3$, then $n|6$.

let the even number $n = 2k$ for some $k \in \mathbb{Z}$. further, we must write that if $n|3$, then $n = 3q$ for some $q \in \mathbb{Z}$. Therefore, for n to be even <u>and</u> divisible by 3, then $n = 2 \cdot 3 \cdot kq$. Say that $kq = x$ for $x \in \mathbb{Z}$, $n = 6x$. by definition of divisibility, $n|6$

since $n|6$ if n is even and $n|3$, and if $n|3$ and n is even then $n|6$, we can conclude that $n|6$ iff n is even and $n|3$. 

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10) let $x \in \mathbb{R}$. Show that $x^2=1$ iif $x=-1$ or $x=1$

if $x^2=1$

$x^2-1=0$

$(x-1)(x+1)=0$

the product of 2 numbers equals zero iif one or both of those numbers equals zero.

therefore, in this case, $x-1=0$ or $x+1=0$.

by this case, $x=1$ or $x=-1$. Therefore,

$x^2=1$ iif $x=-1$ or $x=1$.

B(ii) 5-7

5) $b\mathbb{Z}+r = bq+r$ for some $r \in \mathbb{Z}$

use the strategies in B(ii)-2 to proove

$$\mathbb{Z} = 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$$

(note $3\mathbb{Z} = 3\mathbb{Z}+0$)

B(ii)-2: $A=B$

a. Show $A \subseteq B$ - let $x \in A$ .... $\therefore x \in B$

b. Show $B \subseteq A$ - let $x \in B$ .... $\therefore x \in A$

$\therefore A \subseteq B$ and $B \subseteq A$ so $A=B$.

division algorithm: given an integer m and positive b, there are unique integers q and r with $m=bq+r$ and $0 \leq r \leq b$

(next page!)

Theorem: $\mathbb{Z} = 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$

Proof: 1) first we must show that $\mathbb{Z} \subseteq 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$. assume $x \in \mathbb{Z}$. By the division algorithm, we can say that any $x \in \mathbb{Z}$ can be described as $x = 3q + r$ where $0 \leq r < 3$ and r is some number $r \in \mathbb{Z}$. Therefore, $x \in 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$ by the division algorithm. Therefore, if $x \in \mathbb{Z}$ and $x \in 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$, we can say $\mathbb{Z} \subseteq 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$

2) now, we must proove that $3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2 \subseteq \mathbb{Z}$. Because we know that the sum and product of some integers is an integer we can assume that because $3, 2, 1 \in \mathbb{Z}$, we can assume $3\mathbb{Z} \in \mathbb{Z}$, $3\mathbb{Z}+1 \in \mathbb{Z}$, and $3\mathbb{Z}+2 \in \mathbb{Z}$. So, if $x \in 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$, we can assume $x \in \mathbb{Z}$. Therefore, $3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2 \subseteq \mathbb{Z}$.

So, by definition of equals, because $\mathbb{Z} \subseteq 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$, and $3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2 \subseteq \mathbb{Z}$, we can say that

$$\mathbb{Z} = 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2.$$

b) No, the union $3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$ is not disjoint. this is because any number divided by another can only have one remainder which is contained in $0 \leq r \leq b$ as stated in the division algorithm. No number can have more than one remainder. So, since the remainder which comes from dividing by 3 is $0 \leq r < 3$, $3\mathbb{Z}$ is all the numbers with remainder 0, $3\mathbb{Z}+1$ with remainder 1, $3\mathbb{Z}+2$ with remainder 2.

7) <u>Theorem</u>: for any sets $X, Y,$ and Z,

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

<u>Proof</u>: We must proove by cases,

1) Lets proove that $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$.
First assume that $\ell \in X \cup (Y \cap Z)$. Therefore, $\ell \in X$ or $\ell \in Y \cap Z$. We can also say $\ell \in Y$ and $\ell \in Z$. Because $\ell \in X$ or $\ell \in Y$ and also $\ell \in Z$, we can say $\ell \in (X \cup Y)$ and $\ell \in (X \cup Z)$ by definition of union. Therefore, by definition of intersection it follows that $\ell \in (X \cup Y) \cap (X \cup Z)$. So, because $\ell \in X \cup (Y \cap Z)$ and also $\ell \in (X \cup Y) \cap (X \cup Z)$, we can say $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$.

2) Now we proove that $(X \cup Y) \cap (X \cup Z) \subseteq X \cup (Y \cap Z)$. Assume $\ell \in (X \cup Y) \cap (X \cup Z)$. Therefore $\ell \in (X \cup Y)$ and $\ell \in (X \cup Z)$. So, $\ell \in X$ or $\ell \in Y$ and $\ell \in Y$ By definition of intersection and union, we can say $\ell \in X \cup (Y \cap Z)$. Therefore, if $\ell \in (X \cup Y) \cap (X \cup Z)$ and $\ell \in X \cup (Y \cap Z)$, we say $(X \cup Y) \cap (X \cup Z) \subseteq X \cup (Y \cap Z)$.

Therefore, if $(X \cup Y) \cap (X \cup Z) \subseteq X \cup (Y \cap Z)$ and $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$, we write that

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z).$$ 

TBoP - chap 8 (171) ex. 14, 16

ex. 14) If A, B, and C are sets, then $(A \cup B) - C = (A - C) \cup (B - C)$

Proof: we must consider this by cases.

1) first we proove $(A \cup B) - C \subseteq (A - C) \cup (B - C)$. Assume $x \in (A \cup B) - C$. So, $x \in (A \cup B)$ and $x \notin C$. We have now that $x \in A$ or $x \in B$. We will consider sepcrately

a) let $x \in A$. Because $x \notin C$, we say $x \in A - C$ or $x \in B - C$ By definition of union, $x \in (A - C) \cup (B - C$.

b) Now let $x \in B$, because $x \notin C$, we have that $x \in B - C$ or $x \in A - C$. By definition of union, $x \in (B - C) \cup (A - C)$.

Therefore, because $x \in (A \cup B) - C$ and $x \in (A - C) \cup (B - C)$, we can say $(A \cup B) - C \subseteq (A - C) \cup (B - C)$.

2) Now we must proove $(A - C) \cup (B - C) \subseteq (A \cup B) - C$, Assume $x \in (A - C) \cup (B - C)$. We consider 2 cases:

a) If $x \in A - C$, then $x \in A$ and $x \notin C$. Since $x \in A$, we can say $x \in A$ or $x \in B$. So by definition of union, $x \in (A \cup B)$. Because $x \notin C$, we have $x \in (A \cup B) - C$.

b) If $x \in B - C$, then $x \in B$ and $x \notin C$. Since $x \in B$, we can say $x \in B$ or $x \in A$. So by definition of union, $x \in (A \cup B)$ but $x \notin C$ so $x \in (A \cup B) - C$.

Therefore, because $x \in (A - C) \cup (B - C)$ and $x \in (A \cup B) - C$, we write $(A - C) \cup (B - C) \subseteq (A \cup B) - C$.

Since $(A \cup B) - C \subseteq (A - C) \cup (B - C)$ and $(A - C) \cup (B - C) \subseteq (A \cup B) - C$ we can conclude that $(A \cup B) - C = (A - C) \cup (B - C)$.

ex. 16) <u>Theorem</u>: If A, B, and C are sets, then

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

To proove equality, we must proove that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$, and that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ we will consider 2 cases.

1) first we will prove that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. let $x \in A \times (B \cup C)$. This means $x \in A \times B$ or $x \in A \times C$. So, by definition of union, $x \in (A \times B) \cup (A \times C)$. Because $x \in A \times (B \cup C)$ and $x \in (A \times B) \cup (A \times C)$, we say $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

2) Now lets proove $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. let $x \in (A \times B) \cup (A \times C)$. By definition, $x \in (A \times B)$ or $x \in (A \times C)$. by definition of union, $x \in A \times (B \cup C)$. Because $x \in (A \times B) \cup (A \times C)$ and $x \in A \times (B \cup C)$, we can say $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Since $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ and $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$, we write that

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$