Anina Dragan

Homework #3

5-POP B(i)-4 <u>Thm.</u> If n is an even number, then $n-1$ is an odd number

<u>Proof.</u> (Contraposition).

Assume $n-1$ is not an odd number. Thus, $n-1$ must be even, so $n-1=2k$ for some $k \in \mathbb{Z}$. Therefore, $n=2k+1$, so by the definition of odd, n is odd. It then follows that n is not even. So if $n-1$ is not odd, then n is not even, which shows that if n is even, then $n-1$ is odd, via contraposition. □

B(i)-5 <u>Thm.</u> If m^2 is an odd number, then m is an odd number. □

<u>Proof.</u> (Direct)

Suppose m^2 is odd. Hence, $m^2 = 2\lambda+1$ for some $\lambda \in \mathbb{Z}$. It is also the case that $m = 2k+r$ for some $k \in \mathbb{Z}$ where r equals either 0 or 1. Thus, $m^2 = (2k+r)^2 = 4k^2 + 4kr + r^2$, which equals $2\lambda+1$, or m^2. Suppose that $r=0$, then $4k^2 = 2\lambda+1$, so $m^2 = 2(2k^2) = 2n$, but this would contradict the initial assumption that m^2 is odd, so $r \neq 0$ & must therefore $=1$. Thus, $m = 2k+1$, so m is odd, by definition. Hence, if m^2 is odd, then m is odd as well.

B(i)-6 <u>Thm.</u> If n is not divisible by 4, then n is not divisible by 12.

<u>Proof.</u> (Contraposition)

Suppose it is not the case that n is not divisible by 12. It then follows that n is divisible by 12, expressed as $12 \mid n$. By definition of divides, $n = 12m$ for some $m \in \mathbb{Z}$. Thus, $n = 4 \cdot (3 \cdot m) = 4k$, where $k = 3m$. So it can be said that $4 \mid n$, also by definition of divides. So if $12 \mid n$, it implies that $4 \mid n$, the contrapositive of which says that if n is not divisible by 4, then n is not divisible by 12. □

B(i)-7 <u>Thm.</u> If x is odd, then x is divisible by 3.

<u>Proof.</u> (Counterexample)

Let x be an odd number. So, it could be the case that $x = 7$, and $3 \nmid 7$. Thus, x being odd does not imply that x is divisible by 3. □

B(i)-8 <u>Thm.</u> Let $a, b, c \in \mathbb{Z}$. If $a \mid (b+c)$, then $a \mid b$ and $a \mid c$.

<u>Proof.</u> (Counterexample)

Consider $a, b, c \in \mathbb{Z}$. Thus, a could equal 2, b could equal 7, and c could equal 3, and it would satisfy the condition $a \mid (b+c)$, as $2 \mid (3+7)$. However, it is not the case that $2 \mid 7$ and $2 \mid 3$. So $a \mid (b+c)$ does not imply that $a \mid b$ and $a \mid c$. □

B(i)-9 Thm. $n \in \mathbb{Z}$ is divisible by 6 iff n is both even and divisible by 3.

Proof. (1) Consider $n \in \mathbb{Z}$ such that n is divisible by 6. It follows that $n = 6k$ for some $k \in \mathbb{Z}$. Thus, $n = 2(3k) = 2m$, where m is the value $3k$. So n is even, by definition. It is also the case that $n = 3(2k) = 3b$, where $b = 2k$, and therefore $3|n$, by definition of divides. Hence, if $n \in \mathbb{Z}$ is divisible by 6, then n is even and $3|n$, as required.

(2) Now assume $n \in \mathbb{Z}$ is even and divisible by 3. Thus, $n = 3q$ for some $q \in \mathbb{Z}$, by definition of divides. Because n is even, it must be the case that $3q$ is also even, that is $3q = 2t$ for some $t \in \mathbb{Z}$. So $2|3q$ and thus $2|n$. Because $2|n$ and $3|n$, $(2 \cdot 3)|n$, so n is divisible by 6. Therefore if $n \in \mathbb{Z}$ is even + divisible by 3, it follows that n is divisible by 6, as required.

Because $n \in \mathbb{Z}$ being divisible by 6 implies that n is both even and $3|n$, and $n \in \mathbb{Z}$ being both even + divisible by 3 implies that $6|n$, we conclude that $6|n$ if and only if n is both even and $3|n$ for $n \in \mathbb{Z}$.

B(i)-10 Thm. For $x \in \mathbb{R}$, $x^2 = 1$ iff $x = -1$ or $x = 1$. $\sqcap$

Proof. $x^2 = 1$ iff $x^2 - 1 = 0$, which is true iff $(x+1)(x-1) = 0$, which is true iff $x = -1$ or $x = 1$. So $x^2 = 1$ iff $x = -1$ or $x = 1$.

S-POP B(ii)-5. Thm $\mathbb{Z} = 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$

Proof. (1) We show $\mathbb{Z} \subseteq 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$: let $q \in \mathbb{Z}$. For $b, q, r \in \mathbb{Z}$, $b\mathbb{Z}+r$ denotes the set of all integers of form $bq+r$. If we allow $b=3$, then $3|q$, by definition of divides. If arbitrary integer q is divisible by 3 with no remainder, then $q \in 3\mathbb{Z}$, the set of all integers divisible by 3. By definition of union, $q \in 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$. Thus, $\mathbb{Z} \subseteq 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$.

(2) We show $3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2) \subseteq \mathbb{Z}$: let $p \in 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$. By definition of union $p \in 3\mathbb{Z}$, $p \in 3\mathbb{Z}+1$, or $p \in 3\mathbb{Z}+2$. In all 3 of these cases p is in a set of only integers, so $p \in \mathbb{Z}$, and by definition of a subset, $3\mathbb{Z} \subseteq \mathbb{Z}$, $3\mathbb{Z}+1 \subseteq \mathbb{Z}$, and $3\mathbb{Z}+2 \subseteq \mathbb{Z}$. Thus, by definition of a union, $3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2 \subseteq \mathbb{Z}$.

As we can see, both sets are subsets of each other, so $\mathbb{Z} = 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$.

B(ii)-6 The union is disjoint because none of the 3 sets have any elements in common. Given that the division algorithm says that for $m \in \mathbb{Z}$ and $b \in \mathbb{N}$, there exists $q, r \in \mathbb{Z}$ such that q and r are unique, $m = bq + r$, and $0 \leq r < b$. In this union, $b = 3$, and $0 \leq r = 0, 1, 2 \leq 3$, so the conditions are satisfied for each set in the union to include unique integers, so none of their elements are the same.

B(ii)-7 Thm. $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.

Proof. (1) Let $a \in X \cup (Y \cap Z)$. By definition of union, $a \in X$ or $a \in (Y \cap Z)$. Consider both cases:

(a) If $a \in X$, then by definition of union, $a \in X \cup Y$ and $a \in X \cup Z$. So by definition of intersection, $a \in (X \cup Y) \cap (X \cup Z)$.

(b) If $a \in (Y \cap Z)$ then by definition of intersection $a \in Y$ and $a \in Z$. It follows by definition of union that $a \in X \cup Y$ and $a \in X \cup Z$. Thus, by definition of intersection once again, $a \in (X \cup Y) \cap (X \cup Z)$.

Because in either case, it follows from $a \in X \cup (Y \cap Z)$ that $a \in (X \cup Y) \cap (X \cup Z)$, we conclude $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$.

(2). Let $b \in (X \cup Y) \cap (X \cup Z)$. By definition of intersection, $b \in (X \cup Y)$ and $b \in (X \cup Z)$. It follows that $b \in X$ or $b \in Y$, and $b \in X$ or $b \in Z$. Thus, either $b \in X$ or $b \in Z$ and $b \in Y$.

(a) If $b \in X$, then $b \in X \cup (Y \cap Z)$ by definition of union.

(b) If $b \in Z$ and $b \in Y$, it follows that $b \in Y \cap Z$, by definition of intersection. Thus, $b \in X \cup (Y \cap Z)$, by definition of union.

In either case, we deduce that $b \in X \cup (Y \cap Z)$. Therefore, we conclude that $(X \cup Y) \cap (X \cup Z) \subseteq X \cup (Y \cap Z)$.

Because $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$, and $(X \cup Y) \cap (X \cup Z) \subseteq X \cup (Y \cap Z)$, we can say that $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.

T-BoP

14. Thm. If A, B, and C are sets, then $(A \cup B) - C = (A-C) \cup (B-C)$.
Proof. (1) Let A, B, C be sets and let $x \in (A \cup B) - C$. It follows that $x \in (A \cup B)$ but $x \notin C$. By definition of union $x \in A$ or $x \in B$, let's examine both cases:

(a) If $x \in A$, then because we know $x \notin C$, $x \in (A-C)$ by definition of set difference. By definition of union, $x \in (A-C) \cup (B-C)$.

(b) If $x \in B$, then because we know $x \notin C$, $x \in (B-C)$, by definition of set difference. Hence, $x \in (A-C) \cup (B-C)$, by definition of union.

In either of the two cases, $x \in (A-C) \cup (B-C)$. Thus, we can say $(A \cup B) - C \subseteq (A-C) \cup (B-C)$.

(2) Let A, B, C be sets. Let $y \in (A-C) \cup (B-C)$. By definition of union, $y \in (A-C)$ or $y \in (B-C)$. Examining both possibilities:

(a) If $y \in (A-C)$, then it follows that $y \in A$ and $y \notin C$, by definition of set difference. Because $y \in A$, by definition of union, $y \in A \cup B$. And since we know $y \notin C$, it follows that $y \in (A \cup B) - C$, by definition of set difference.

(b) If $y \in (B-C)$, then by definition of set difference, $y \in B$ and $y \notin C$. It follows from $y \in B$ that $y \in A \cup B$ by definition of union. Because $y \notin C$ has been established, we conclude $y \in (A \cup B) - C$, by definition of set difference.

In either case, $y \in (A \cup B) - C$. Since this follows from $y \in (A-C) \cup (B-C)$, we can say that $(A-C) \cup (B-C) \subseteq (A \cup B) - C$.
Because both sets are subsets of each other, we can conclude that $(A \cup B) - C = (A-C) \cup (B-C)$.

16. Thm. If A, B, C are sets, then $A \times (B \cup C) = (A \times B) \cup (A \times C)$
Proof. (1) Let A, B, C be sets and let $n \in A \times (B \cup C)$. By definition of the cartesian product, $n = (x, y)$ such that (x, y) is an ordered pair with $x \in A$, $y \in B \cup C$. By definition of a union,

$y \in B$ or $y \in C$:

(a) If $y \in B$, then because $x \in A$, it follows that $n \in (A \times B)$ by definition of the cartesian product. Further, by definition of union, $n \in (A \times B) \cup (A \times C)$.

(b) If $y \in C$, $n \in (A \times C)$ by definition of the cartesian producted, as the other part of the ordered pair, $x \in A$. Thus, by definition of union, $n \in (A \times B) \cup (A \times C)$.

In either case, it follows that $n \in (A \times B) \cup (A \times C)$. Therefore, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

(2). Let A, B, C be sets such that $m \in (A \times B) \cup (A \times C)$. By definition of union, $m \in (A \times B)$ or $m \in (A \times C)$. Let m be ordered pair (q, p).

(a) If $m \in (A \times B)$, it follows from the definition of the cartesian product that $q \in A$ and $p \in B$. If $p \in B$, then by definition of union, $p \in B \cup C$. So, $m \in A \times (B \cup C)$.

(b) If $m \in (A \times C)$, then by definition of cartesian product, $q \in A$, $p \in C$. Therefore, $p \in (B \cup C)$ by definition of union. Thus, $m \in A \times (B \cup C)$, by definition of cartesian product.

Because $m \in A \times (B \cup C)$ in both cases, we can conclude that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Finally, because both sets are subsets of each other, it follows that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.