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Homework 3

T-BOP (4) Proposition: If A, B, C are sets, then $(A \cup B) - C = (A-C) \cup (B-C)$

Proof. Let $A, B,$ and C be sets.

Assume $x \in (A \cup B) - C$.

Then by definition, $x \in A$ or $x \in B$ and $x \notin C$

So $x \in A$ and $x \notin C$ or $x \in B$ and $x \notin C$

Thus by def'n, $x \in (A-C) \cup (B-C)$

Therefore $(A \cup B) - C = (A-C) \cup (B-C)$ $\square$

16) Proposition: If $A, B,$ and C are sets, then $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Proof. Let $A, B,$ and C be sets

1) Assume $x, y \in A \times (B \cup C)$

Then by def'n, $y \in B$ or $y \in C$ and $x \in A$

Thus $x \in A$ and $y \in B$ or $x \in A$ and $y \in B$

Therefore $x, y \in (A \times B) \cup (A \times C)$

2) Assume $x, y \in (A \times B) \cup (A \times C)$

Then by def'n $x \in A$ and $y \in B$ or $x \in A$ and $y \in C$

Thus $x \in A$ and $y \in B$ or $y \in C$

Therefore $A \times (B \cup C)$

So $A \times (B \cup C) = (A \times B) \cup (A \times C)$ $\square$

S-POP B(i)-4)

Proposition: If n is an even number then $n-1$ is an odd number

Proof: Assume $n-1$ is not odd

So $n-1$ is even

So $n-1 = 2k$ for some $k \in \mathbb{Z}$

Thus $n = 2k+1$ for some $k \in \mathbb{Z}$

Then by definition of odd, n is odd

So n is not even

Therefore if n is even, $n-1$ is odd $\square$

B(i)-5)

Proposition: If m^2 is an odd number, then m is an odd number.

Proof: Assume m^2 is odd

Thus $m^2 = 2l+1$ where $l \in \mathbb{Z}$

So $m = 2k+r$ where $k \in \mathbb{Z}$ and $r=0$ or $r=1$

Thus $4k^2+4kr+r^2 = 2l+1$.

So $2k^2+2kr = \frac{2l+1-r^2}{2}$

Because $2k^2+2kr$ has a factor of two, $2k^2+2kr$ is an even integer

Since m^2 is odd, $2l+1-r^2$ must be odd, so r^2 is even.

When $r=0$, $r^2=0$ which is even but if $r=1$, $r^2=1$ which is odd, and contridicts the assumption that m^2 is odd.

So m must be odd

Therefore, if m^2 is an odd number, then m is an odd number. $\square$

B(i)-6) Proposition: If n is not divisible by 4, then n is not divisible by 12

Proof. Assume n is divisible by 12

Thus $n = 12k$ $k \in \mathbb{Z}$

So $n = (4 \cdot 3)k$ $k \in \mathbb{Z}$

Then $n = 4(3k)$ where $3k \in \mathbb{Z}$ since $k \in \mathbb{Z}$

Therefore $n = 4m$ where $m = 3k = \mathbb{Z}$

So n is divisable by 12 and therefore divisible by 4

Therefore if n is not divisible by 4, then n is not divisable by 12 $\square$

B(i)-7) Proposition: If x is odd, then x is divisable by 3

False. 5 is odd and not divisable by 3.

B(i)-8) Proposition: If $a|(b+c)$ then $a|b$ and $a|c$

Proof. Assume $a|(b+c)$

Thus $(b+c) = ak$ where $k \in \mathbb{Z}$

So, $b = ak - c$ and $c = ak - b$

Thus ak is a multiple of k and b and c are differences involving ak

Therefore $a|b$ and $a|c$ $\square$

B(i)-9) Proposition: An integer n is divisible by 6 if and only if n is both even and divisible by 3

Proof.

1) Assume n is divisible by 6 and $n \in \mathbb{Z}$

Thus $n = 6k$ where $k \in \mathbb{Z}$

Then $n = (3 \cdot 2)k = 2(3k)$ $k \in \mathbb{Z}$

Therefore n is even and divisible by 3

2) Assume n is an even integer and divisible by 3

Thus $n = 2m$ and $n = 3p$ where $m \in \mathbb{Z}$ and $p \in \mathbb{Z}$

Then $2m = 3p$

Since $2m$ must be even, $3p$ must be even.

So $2m = 3(2g)$ where $2g = p$ and $g \in \mathbb{Z}$

So $2m = 6g = m = 3g$

Therefore n is divisible by 6.

So: an integer n is divisible by 6 if and only if n is both even and divisible by 3 and an integer n is both even and divisible by 3 if and only if n is divisible by 6. $\square$

B(i)-10) Proposition: $x^2=1$ if and only if $x=-1$ or $x=1$

Proof. $x^2=1$ iff $x^2-1=0$ which is true iff $(x-1)(x+1)=0$ which is true if $(x+1)=0$ or $(x-1)=0$ which is true iff $x=-1$ or $x=1$. Therefore, $x^2=1$ iff $x=1$ or $x=-1$. $\square$

S-POP B(ii)-5)

Proposition: $\mathbb{Z} = 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$

Proof. a) Assume $x \in \mathbb{Z}$.

By the division algorithm $x = 3q + r$ where $r = 0$, or $r = 1$, or $r = 2$

Thus $x = 3\mathbb{Z} + 0$ if $r = 0$, $x = 3\mathbb{Z}+1$ if $r = 1$ and $x = 3\mathbb{Z}+2$ if $r = 2$

Thus $x \in 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$

Therefore $\mathbb{Z} \subseteq 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$

b) Assume $x \in 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$

By the division algorithm $x = 3q + r$ where $r = 0$, or $r = 1$ or $r = 2$

Since $q \in \mathbb{Z}$, $3q$ is divisible by 3 so x is divisible by 3

Thus by definition of union, if $r = 0$, then x is in $3\mathbb{Z}$, if $r = 1$, then x is in $3\mathbb{Z}+1$, if $r = 2$, then x is in $3\mathbb{Z}+2$.

So $x \in \mathbb{Z}$

Therefore $3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2) \subseteq \mathbb{Z}$

Since $\mathbb{Z} \subseteq 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$ and $3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2) \subseteq \mathbb{Z}$

$\mathbb{Z} = 3\mathbb{Z} \cup (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$ $\square$

B(ii)-6) The sets $3\mathbb{Z}$, $3\mathbb{Z}+1$, and $3\mathbb{Z}+2$ do not have any elements in common. This is due to the unique integers q and r in the division algorithm $m = bq + r$ where $0 \leq r < |b|$. In this case, the remainder is either 0, 1, or 2 so an integer cannot satisfy all the conditions for two sets.

B(ii)-7) Proposition: $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

Proof. a) $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$

Assume $a \in X \cup (Y \cap Z)$

By definition of union, $a \in X$ or $a \in (Y \cap Z)$

Thus if $a \in X$ then $a \in (X \cup Y)$ and $a \in (X \cup Z)$

Then if $a \in (Y \cap Z)$ then by definition of intersect, $a \in Y$ and $a \in Z$

So $a \in (X \cup Y) \cap (X \cup Z)$ in either case

Therefore $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$

b) $(X \cup Y) \cap (X \cap Z) \subseteq X \cup (Y \cap Z)$

Assume $a \in (X \cup Y) \cap (X \cap Z)$

By definition $a \in (X \cup Y)$ and $a \in (X \cap Z)$

Thus $a \in X$ and $a \in Y$ or Z by definition of union

So $a \in X$ then $a \in X \cup (Y \cap Z)$

Therefore $(X \cup Y) \cap (X \cup Z) \subseteq X \cup (Y \cap Z)$

Since $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$ and $(X \cup Y) \cap (X \cup Z) = X \cup (Y \cap Z)$,

$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ □