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HW 3

S-POP

B(i)

4.) Proposition: If n is an even number, then $n-1$ is an odd number.

Proof (contraposition):

Assume $n-1$ is not an odd number.
This means that $n-1$ is even.
We may write $n-1 = 2k$ for some integer k.
But then $n = 2k+1$, so n is odd.
Therefore, n is not an even number.
$\sim Q \rightarrow \sim P \Rightarrow P \rightarrow Q$. $\square$

5.) Proposition: If m^2 is an odd number, then m is an odd number.

Proof:

Assume m^2 is odd.
Then we have $m^2 = 2l+1$ for $l \in \mathbb{Z}$, by def'n of an odd number.
Now let's assume $m = 2k+r$ for $k \in \mathbb{Z}$ and $r=0$ or $r=1$
via the divisibility algorithm.
If $m = 2k+r$, then $m^2 = (2k+r)^2$.
Since l and k are arbitrary integers, we can substitute n for both.
So we have $m^2 = 2n+1$ and $m^2 = (2n+r)^2$.
Setting them equal, we get $2n+1 = (2n+r)^2$

$$2n+1 = 4n^2 + 4nr + r^2$$

6.) Proposition: If n is not divisible by 4, then n is not divisible by 12.

Proof (contraposition):

Let's assume n is divisible by 12.
Then there exists an integer k such that $n = 12k$.
Since $n = 12k$, we can write $n = (3 \cdot 4)k$

$$= 3(4k).$$

If we let $m = 3k$, then $n = 4m$.
Thus, n is divisible by 4.
$\sim Q \rightarrow \sim P$. QED

7.) Proposition: If x is odd, then x is divisible by 3.

Counterexample: 5 is odd but not divisible by 3.

8.) Statement: If $a \mid b$ and $a \mid c$, then $a \mid (b+c)$.

Converse: If $a \mid (b+c)$, then $a \mid b$ and $a \mid c$.

Counterexample: $a = 5$, $b = 2$, $c = 3$

$$5 \mid (2+3), \text{ but } 5 \nmid 2 \text{ and } 5 \nmid 3.$$

9.) Proposition: An integer n is divisible by 6 iff n is both even and $3 \mid n$.

Proof:

a) To show that $P \Rightarrow Q$:

Let's assume $6 \mid n$ for $n \in \mathbb{Z}$.

If $6 \mid n$, then $n = 6k$ for some integer k.

This means that $n = 2 \cdot 3 \cdot k$

$= 2 \cdot 3k$.

This is of the form $2x$ for some $x \in \mathbb{Z}$, so it is even.

It is also of the form $3 \cdot y$ for some $y \in \mathbb{Z}$, so $3 \mid n$.

Therefore, if $6 \mid n$, then n is even and $3 \mid n$.

$P \Rightarrow Q$.

b.) To show that $Q \Rightarrow P$:

Let's assume n is both even and $3 \mid n$.

Since $3 \mid n$, then $n = 3k$ for $k \in \mathbb{Z}$.

If n is even and $n = 3k$, this means that k must be an even number.

So $n = 3(2\ell)$ for $k = 2\ell$.

Algebraically, we get $n = 3(2\ell) = 6\ell$.

Therefore, n is divisible by 6.

$Q \Rightarrow P$.

Since $P \Rightarrow Q$ and $Q \Rightarrow P$, we get $P \Leftrightarrow Q$. QED

10.) Proposition: $x^2 = 1$ iff $x = -1$ or $x = 1$.

Proof: $x^2 = 1$ iff $x = (-1)^2$ or $x = (1)^2$

B(ii)

5.) Proposition: $\mathbb{Z} = 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$

Proof:

a.) we show that $\mathbb{Z} \subseteq 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$:

Let $x \in \mathbb{Z}$. $\qquad m = bq + r$ and $0 \le r < b$

There exists an integer x where $x = 3\mathbb{Z} + r$ and $0 \le r < 3$ via the division algorithm.

This means that $x \in 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$.

So, $\mathbb{Z} \subseteq 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$.

b.) we show that $3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2 \subseteq \mathbb{Z}$.

Let $x \in 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$.

Since this is of the form $n = 3\mathbb{Z} + r$ and $0 \le r < 3$,

$n = \mathbb{Z}$.

Therefore, $x \in \mathbb{Z}$.

This means that $3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2 \subseteq \mathbb{Z}$.

$A \subseteq B$ and $B \subseteq A$.

$\mathbb{Z} = 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$. QED.

6.) The sets have no elements in common, so they are disjoint. I reached this conclusion by inputting consecutive integer values for $\mathbb{Z}$ and comparing the results. there is no overlap.

7.) Proposition: $\underbrace{X \cup (Y \cap Z)}_{A} = \underbrace{(X \cup Y) \cap (X \cup Z)}_{B}$

a.) $A \subseteq B$:

Let $x \in X \cup (Y \cap Z)$.

This means that $x \in X$ or $x \in (Y \cap Z)$, by def'n of union.

cases 1.) If $x \in X$, then $x \in X \cup Y$ and $x \in X \cup Z$ by def'n of union.

therefore, $x \in (X \cup Y) \cap (X \cup Z)$.

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2.) If $x \in Y \cap Z$, then $x \in Y$ and $x \in Z$ by def'n of intersection.
Thus, $x \in (X \cup Y) \cap (X \cup Z)$.
In both cases, $x \in (X \cup Y) \cap (X \cup Z)$.
So $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$.

b.) $B \subseteq A$:
Let $x \in (X \cup Y) \cap (X \cup Z)$.
This means $x \in (X \cup Y)$ and $x \in (X \cup Z)$ by def'n of intersection

cases
1.) If $x \in X$, then $x \in X \cup (Y \cap Z)$, by def'n of union
2.) If $x \in Y$ and $x \notin X$, then $x \in Y \cap Z$ by def'n of intersection.
Thus $x \in X \cup (Y \cap Z)$.
3.) If $x \in Z$ and $x \notin X$, then $x \in Y \cap Z$.
Thus $x \in X \cup (Y \cap Z)$.
In all cases, $x \in X \cup (Y \cap Z)$.
So, $(X \cup Y) \cap (X \cup Z) \subseteq X \cup (Y \cap Z)$.

Since $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$ and $(X \cup Y) \cap (X \cup Z) \subseteq X \cup (Y \cap Z)$, it follows that $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$. $\square$

Ch. 8 TBOP

14. Proposition: $\underbrace{(A \cup B) - C}_{A} = \underbrace{(A - C) \cup (B - C)}_{B}$.

Proof:

a.) $A \subseteq B$:
Let $x \in (A \cup B) - C$.
This means that $x \in A$, $x \in B$, and $x \notin C$ by def'n of union and difference.

cases
1.) If $x \in A$, then $x \in (A - C) \cup (B - C)$.
2.) If $x \in B$, then $x \in (A - C) \cup (B - C)$.
Thus $(A \cup B) - C \subseteq (A - C) \cup (B - C)$.

b.) $B \subseteq A$:
Let $x \in (A - C) \cup (B - C)$.
This means that $x \in (A - C)$ and $x \in (B - C)$.

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If $x \in (A-C)$ and $x \in (B-C)$, then it follows that $x \in (A \cup B) - C$.

In all cases, $(A \cup B) - C \subseteq (A-C) \cup (B-C)$ and $(A-C) \cup (B-C) \subseteq (A \cup B) - C$.

Therefore $(A \cup B) - C = (A-C) \cup (B-C)$. QED.

16.) Proposition: $\overbrace{A \times (B \cup C)}^{A} = \overbrace{(A \times B) \cup (A \times C)}^{B}$.

Proof:

a.) $\boxed{A \subseteq B:}$

Let $(x,y) \in A \times (B \cup C)$.

So $x \in A$ and $y \in (B \cup C)$.

1.) If $y \in B$, then (x,y) is in $(A \times B)$.

Thus, $(x,y) \in (A \times B) \cup (A \times C)$.

2.) If $y \in C$, then (x,y) is in $(A \times C)$.

In both cases, $(x,y) \in (A \times B) \cup (A \times C)$.

So, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

b.) $\boxed{B \subseteq A:}$

Let $(x,y) \in (A \times B) \cup (A \times C)$.

This means that (x,y) is in either $(A \times B)$ or $(A \times C)$ by def'n of union.

1.) If (x,y) is in $(A \times B)$, then $x \in A$ and $y \in B \cup C$ by def'n of union.

2.) If (x,y) is in $(A \times C)$, then $x \in A$ and $y \in B \cup C$ by def'n of union.

In both cases, $(x,y) \in A \times (B \cup C)$.

So, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Since $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ and $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$, it follows that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.