Homework 1

Bodhi Rubinstein

B(i)-1

a) Theorem: If $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ are even, then $n+m$ is even

Proof: Assume $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ are even. Then by definition, $n = 2k$ for some $k \in \mathbb{Z}$ and $m = 2l$ for some $l \in \mathbb{Z}$. But then, $n+m = 2k+2l = 2(l+k) = 2p$, where $p = l+k$. So $n+m$ is even. $\square$

b) Theorem: If $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ are odd, then $n \cdot m$ is odd.

Proof: Assume $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ are odd. Then by definition, $n = 2k+1$ for some $k \in \mathbb{Z}$ and $m = 2l+1$ for some $l \in \mathbb{Z}$. But then, $n \cdot m = (2k+1)(2l+1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1 = 2p+1$, where $p = (2kl + l + k)$. So $n \cdot m$ is odd.

B(i)-2:

a) Theorem: If $n \in \mathbb{Z}$ is even, then $n^2 | 4$.

Proof: Assume $n \in \mathbb{Z}$ is even.

Then by definition, $n = 2k$ for some $k \in \mathbb{Z}$

Then it follows that $n^2 = (2k)^2 = 4k^2 = 4l$, where $l = k^2$.

Since $n^2 = 4 \cdot l$, by definition $n^2 | 4$.

So for $n \in \mathbb{Z}$ is even, $n^2 | 4$. $\square$

B(i)-3:

a) Theorem: If $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, and $c \in \mathbb{Z}$, then if $a|b$ and $a|c$, then it follows that $a|(b+c)$.

Proof: Assume $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, $c \in \mathbb{Z}$ and that $a|b$ and $a|c$.

Then by definition, $b = an$ for some $n \in \mathbb{Z}$ and $c = am$ for some $m \in \mathbb{Z}$.

Then it follows that $b + c = an + am = a(n+m) = al$, where $l \in \mathbb{Z}$

Then by definition, $a|(b+c)$ since $b+c = al$.

So for $a, b, c \in \mathbb{Z}$ and $a|b$ and $a|c$, $a|(b+c)$

Section 1.1: A(1,3,6,9,14), B(17,18,26,27), C(29,32,34,38)

A) 1) $\{5x-1 : x \in \mathbb{Z}\} = \{\ldots,-16,-11,-6,-1,4,9,14\ldots\}$

3) $\{x \in \mathbb{Z} : -2 \leq x < 7\} = \{-2,-1,0,1,2,3,4,5,6\}$

6) $\{x \in \mathbb{R} : x^2 = 9\} = \{-3,3\}$

9) $\{x \in \mathbb{R} : \sin \pi x = 0\} = \{\ldots -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots\}$

14) $\{5x : x \in \mathbb{Z}, |2x| \leq 8\} = \{-20,-15,-10,-5,0,5,10,15,20\}$

B) 17) $\{2,4,8,16,32,64,\ldots\} = \{2^n : n \in \mathbb{Z}, n \geq 1\}$

18) $\{0,4,16,36,64,100,\ldots\} = \{(2n)^2 : n \in \mathbb{Z}, n \geq 0\}$

26) $\{\ldots,\frac{1}{27},\frac{1}{9},\frac{1}{3},1,3,9,27,\ldots\} = \{3^n : n \in \mathbb{Z}\}$

27) $\{\ldots,-\pi,-\frac{\pi}{2},0,\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi,\frac{5\pi}{2}\} = \{\frac{\pi}{2}n : n \in \mathbb{Z}\}$

C) 29) $|\{\{1\},\{2,\{3,4\}\},\emptyset\}| = 3$

32) $|\{\{\{1,4\}\},a,b,\{\{3,4\}\},\{\emptyset\}\}| = 5$

34) $|\{x \in \mathbb{N} : |x| < 10\}| = |\{0,1,2,3,4,\ldots\}| = 11$

38) $|\{x \in \mathbb{N} : 5x \leq 20\}| = |\{0,5,10,15,20\}| = 5$

Chapter 4: #5, 7, 14

5) Theorem: If $x,y \in \mathbb{Z}$ and x is even, then xy is even.

Proof: Suppose $x,y \in \mathbb{Z}$ and x is even. Then by definition, $x = 2k$ for some $k \in \mathbb{Z}$. then it follows that $x \cdot y = 2k \cdot y = 2ky$. Therefore $xy = 2\ell$ where an integer $\ell = k \cdot y$. So by definition, xy is even. thus if $x,y \in \mathbb{Z}$ and x is even, the xy is even □

7) Theorem: Suppose $a, b \in \mathbb{Z}$. If $a | b$, then $a^2 | b^2$.
Proof: Assume $a, b \in \mathbb{Z}$ and $a | b$.
Then by definition, $b = an$ for some $n \in \mathbb{Z}$.
But then, $b^2 = (an)^2 = a^2 n^2 = a^2 l$ where $l = n^2 = n \cdot n$.
So by definition, $a^2 | b^2$ because $b^2 = a^2 l$ for $l = n^2$, $l \in \mathbb{Z}$
Therefore, if $a, b \in \mathbb{Z}$ and $a | b$, then $a^2 | b^2$. $\square$

14) Theorem: If $n \in \mathbb{Z}$, then $5n^2 + 3n + 7$ is odd.
Proof: Assume $n \in \mathbb{Z}$
We consider two cases:
Case 1: n is even. Then by definition, $n = 2k$ for some $k \in \mathbb{Z}$.
It follows that $5n^2 + 3n + 7 = 5(2k)^2 + 3(2k) + 7 = 20k^2 + 6k + 7$
$= 20k^2 + 6k + 6 + 1 = 2(10k^2 + 6k + 3) + 1 = 2l + 1$, where $l = 10k^2 + 6k + 3$.
Then by definition, $5n^2 + 3n + 7$ is odd.
Case 2: n is odd. Then by definition, $n = 2k + 1$ for some $k \in \mathbb{Z}$.
It follows that $5n^2 + 3n + 7 = 5(2k+1)^2 + 3(2k+1) + 7 = 20k^2 + 20k + 5 + 6k + 3 + 7$
$= 20k^2 + 26k + 15 = 20k^2 + 26k + 14 + 1 = 2(10k^2 + 13k + 7) + 1 = 2l + 1$,
where $l = 10k^2 + 13k + 7$. So $5n^2 + 3n + 7$ is odd.

Since $n \in \mathbb{Z}$ must be even or odd, it follows that $5n^2 + 3n + 7$ is always odd. $\square$