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1/23/24

Homework 1
MATH 2001-004

S-POP Part B(i)

Exercise 1

(a) Theorem. The sum of two odd numbers is even.

Proof.

1. Assume n is an odd number
2. Then, by definition, $n = 2k+1$ for some $k \in \mathbb{Z}$.

So $n+n$ is equivalent to $(2k+1)+(2k+1)$.

$(2k+1)+(2k+1) = 2(2k+1)$

$2(2k+1) = 2l$ where $l = 2k+1$

So, $n+n = 2l$

So, by definition, $n+n$ is even.

So, if n is odd, the sum of two odd numbers n and n is even. $\square$

(b) Theorem. The product of two odd numbers is odd.

Proof.

Assume n is an odd number.

Then, by definition, $n = 2k+1$ for some $k \in \mathbb{Z}$.

So, $n \cdot n = (2k+1)(2k+1)$.

$(2k+1)(2k+1) = 4k^2 + 4k + 1$

$4k^2 + 4k + 1 = 2(2k^2+2k) + 1$

$2(2k^2+2k)+1 = 2l+1$ when $l = 2k^2+2k$

So, $n \cdot n = 2l+1$

So, by definition, the product of $n \cdot n$ is odd.

So, if n is an odd number, the product of two odd numbers n and n is also odd. $\square$

Exercise 2

(a) Theorem. If n is an even number, n^2 is divisible by 4.

Proof.

Let n be an even number.

Then, by definition, $n = 2k$ for some $k \in \mathbb{Z}$.

So, $n^2 = (2k)^2$

$(2k)^2 = 4k^2$ and $4 \mid 4k^2$

Therefore, $4 \mid n^2$ $\square$

(b) Theorem. If n is an odd number, then $n^2 - 1$ is divisible by 4.

Proof

Let n be an odd number.

Then, by definition, $n = 2k+1$ for some $k \in \mathbb{Z}$.

So, $n^2 - 1 = (2k+1)^2 - 1$

$(2k+1)^2 - 1 = 4k^2 + 4k + 1 - 1$

$4k^2 + 4k + 1 - 1 = 4k^2 + 4k$

$4k^2 + 4k = 4(k^2 + k)$

$4(k^2 + k) = 4\ell$ when $\ell = 2k^2 + 2k$

$4 \mid 4\ell$

Since $n^2 - 1 = 4\ell$ and $4 \mid 4\ell$, then $4 \mid (n^2 - 1)$. $\square$

Exercise 3

(a) Theorem. If $a|b$ and $a|c$, then $a|(b+c)$ when $a, b, c \in \mathbb{Z}$.

Proof.

Let $a, b, c \in \mathbb{Z}$ and that $a|b$ and $a|c$.
Then, by definition, $b = al$ where $l \in \mathbb{Z}$.
Then, by definition, $c = at$ where $t \in \mathbb{Z}$.
So, $b + c = al + at$
$al + at = a(l + t)$
$a \mid a(l + t)$
Therefore, $a \mid (b + c)$ $\square$

(b) Theorem. If $a|b$ then $a|bn$ for any $n \in \mathbb{Z}$.

Proof.

Let a, b, n be integers, and $a|b$.
By definition, $b = ac$ where $c \in \mathbb{Z}$
Then, $n \cdot b = n \cdot ac$
$n \cdot c \cdot a = la$ when $l = n \cdot c$
So, $nb = la$
Then, by definition, $a|nb$. $\square$

I-BOP section 1.1

Exercise A

1) $\{\ldots, 5(0)-1, 5(1)-1, 5(2)-1, \ldots\} = \{\ldots -1, 4, 9, \ldots\}$

3) $\{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$

6) $\{-\sqrt{9}, \pm\sqrt{9}\} = \{-3, 3\}$

9) $\arcsin(0) = \pi x$ when $x \in \mathbb{Z}$ $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

14) $|2x| \leq 8 \rightarrow -4 \leq x \leq 4$ $\{-20, -15, -10, -5, 0, 5, 10, 15, 20\}$

Exercise B

17) $\{2n : n \in \mathbb{Z}, n > 0\}$

18) $\{4n^2 : n \in \mathbb{Z}, n \geq 0\}$

26) $\{3^n : n \in \mathbb{Z}\}$

27) $\{\frac{\pi}{2} \cdot n : n \in \mathbb{Z}\}$

Exercise C

29) $|\{\{1\}, \{2\{3,4\}\}, \emptyset\}| = |3|$

32) $|\{\{\{1,4\}, a, b, \{3,4\}\}, \emptyset\}| = |2|$

34) $|\{x \in \mathbb{N} : |x| < 10\}| = |\{1, 2, 3, 4, 5, 6, 7, 8, 9\}| = |9|$

38) $|\{x \in \mathbb{N} : 5x \leq 20\}| = |\{1, 2, 3, 4\}| = |4|$

I-BOP Chapter 4

Problem 5 Theorem. If $x, y \in \mathbb{Z}$ and x is even, then xy is also even.

Proof.

Let x be an even integer, and $y \in \mathbb{Z}$.

Then, by definition, $x = 2k$ for some $k \in \mathbb{Z}$.

So, $xy = 2ky$

$2ky = 2m$ where $m = ky$

So, by definition, m is an even integer.

So, xy is an even integer. $\Box$

Problem 7 Theorem. If $a, b \in \mathbb{Z}$ and $a \mid b$, then $a^2 \mid b^2$.

Proof.

Let $a, b \in \mathbb{Z}$ and $a \mid b$.

By definition, $b = a\ell$ where $\ell \in \mathbb{Z}$

Then, $b^2 = a^2 \ell^2$

$b^2 = a^2 y$ where $y = \ell^2$

So, by definition, $a^2 \mid b^2$. $\Box$

Problem 14 Theorem. If $n \in \mathbb{Z}$, then $5n^2 + 3n + 7$ is odd.

Proof.

Let $n \in \mathbb{Z}$.

Consider two cases:

Case 1: Let n be odd.

By definition, $n = 2k+1$ for some $k \in \mathbb{Z}$.

Then, $5n^2 + 3n + 7 = 5(2k+1)^2 + 3(2k+7) + 7$

$5(2k^2+1)^2 + 3(2k+1) + 7 = 20k^2 + 26k + 15$

$20k^2 + 26k + 15 = 2(10k^2 + 13k + 7) + 1$

$2(10k^2 + 13k + 7) + 1 = 2m + 1$ when $m = 10k^2 + 13k + 7$

By definion, $5n^2 + 3n + 7$ is odd when n is odd.

Case 2: Let n be even

By definition, $n = 2k$ where $k \in \mathbb{Z}$.

Then, $5n^2 + 3n + 7 = 5(2k)^2 + 3(2k) + 7$

$5(2k)^2 + 3(2k) + 7 = 20k^2 + 6k + 7$

$20k^2 + 6k + 7 = 2(10k + 3k + 3) + 1$

$2(10k^2 + 3k + 3) + 1 = 2m + 1$ where $m = 10k^2 + 3k + 3$

So, by definition, $5n^2 + 3n + 7$ is odd when n is even.

Because n may only be either odd or even, then $5n^2 + 3n + 7$ is odd for all $n \in \mathbb{Z}$.