Math 2001 - HW 1

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B(i)-1: Let $n, k \in \mathbb{Z}$. By definition, an odd number is defined as $2n+1$. Adding two odd numbers for some $n, k \in \mathbb{Z}$ results in $(2n+1)+(2k+1)$. We simplify this to $2n+2k+2$. After factoring a 2, the equation becomes $2(n+k+1)$. Since it is known that $2n$ is an even number for some $n \in \mathbb{Z}$, the form $2(n+k+1)$ is even. Thus, the sum of two odd numbers is even. $\square$

b) Let $n, k \in \mathbb{Z}$. By definition, an odd number can be written as $2n+1$. Using n, k, we can multiply two odd numbers using the definition, $(2n+1)(2k+1)$. Simplifying this expression results in $4nk+2n+2k+1$. Factoring out a 2 of the first three terms gives us $2(2nk+n+k)+1$. This factored expression has the form $2n+1$, which makes the result odd. Thus, the product of two odd numbers is odd. $\square$

B(i)-2 (a): Let $n=2k$, $k \in \mathbb{Z}$. Squaring n in terms of k gives us $4k^2$. Dividing $4k^2$ by 4 simply leaves us with k^2. $\left(\frac{4k^2}{4}\right)=k^2$. This proves that 4 divides n^2 with no remainder. Thus, an even number n, after squaring, is divisible by 4. $\square$

b) Let $n = 2k+1$, $k \in \mathbb{Z}$. We can write n^2-1 in terms of k, $(2k+1)^2 - 1 = 4k^2+4k$. Factoring out a 4 from the two terms terms gives us $4(k^2+k)$. Dividing by 4 results in k^2+k, meaning the expression is still an integer. Thus, squaring an odd number and subtracting 1 is divisible by 4. $\square$

B(i)-3 (a): Let $a, b, c \in \mathbb{Z}$. We can write $a|b$ and $a|c$ as $\frac{b}{a}$ and $\frac{c}{a}$ respectively. Since, for both fractions, the divisor is the same (a), we can sum the dividends into one fraction: $\frac{(b+c)}{a}$. This is equal to $a|(b+c)$. Thus, a divides the quantity $b+c$. $\square$

b) Let $a, b, n \in \mathbb{Z}$. Rewriting the expressions as $\frac{b}{a}$ and $\frac{n \cdot b}{a}$, we can simply factor out the integer n from the second expression: $n(\frac{b}{a})$. a still divides b; this quantity is simply scaled by n. Thus, when $a|b$, $a|nb$ is also true for any integer n. $\square$

A1) $\{\ldots -16, -11, -6, -1, 4, 9, 14 \ldots\}$ 3) $\{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$

6) $\{-3, 3\}$ 9) $\{0, 1\}$ 14) $\{-20, -15, -10, -5, 0, 5, 10, 15, 20\}$

B17) $\{2^x : x \in \mathbb{N}\}$ 18) $\{4x^2 : x \geq 0, x \in \mathbb{Z}\}$

26) $\{3^x : x \in \mathbb{Z}\}$ 27) $\{\frac{\pi x}{2} : x \in \mathbb{Z}\}$

C29) $|\{\{1\}, \{2, \{3, 4\}\}, \emptyset\}| = 3$

32) Cardinality $= 4$ 34) $|\{x \in \mathbb{N} : |x| < 10\}| = 9$

38) $|\{x \in \mathbb{N} : 5x \leq 20\}| = 4$

5) Let $x, y \in \mathbb{Z}$. $x = 2n$ for some $n \in \mathbb{Z}$. Writing $x \cdot y$ in terms of n gives us $2n \cdot y$. This can also be written as $2(ny)$. Since an even number can be expressed as $2k$, for some $k \in \mathbb{Z}$, $2(ny)$ is even. Therefore, if x is even, xy is also even. $\square$

7) Let $a, b \in \mathbb{Z}$. Rewriting $a|b$, we get $\frac{b}{a}$, and rewriting $a^2|b^2$, we get $\frac{b^2}{a^2}$. Splitting this expression gives us $\frac{b}{a} \cdot \frac{b}{a}$. a still divides b even when both integers are squared, thus $a^2|b^2$ when $a|b$. $\square$

14) Let $n \in \mathbb{Z}$.

Case 1: n is even. Rewrite n as $2k$ for some $k \in \mathbb{Z}$.
$5(2k)^2 + 3(2k) + 7 = 20k^2 + 6k + 7$. Factoring out a 2 from the first two expressions gives us $2(10k^2 + 3k) + 7$. Since $10k^2 + 3k$ is an integer, and it is multiplied by 2, that term is even. By definition, adding an odd number to an even number is always odd, which makes $5n^2 + 3n + 7$ odd when n is even.

Case 2: n is odd. Rewrite n as $2k+1$ for some $k \in \mathbb{Z}$.
$5(2k+1)^2 + 3(2k+1) + 7 = 5(4k^2 + 4k + 1) + 6k + 10 \ldots$
$\ldots = 20k^2 + 20k + 5 + 6k + 10 = 20k^2 + 26k + 15$. Factoring out a 2 from the first two terms gives us $2(10k^2 + 13k) + 15$. Since $10k^2 + 13k$ is an integer, and it is multiplied by 2, that term is even. Adding 15 makes the whole expression odd. This makes $5n^2 + 3n + 7$ odd when n is odd. In conclusion, when n is either even or odd, $5n^2 + 3n + 7$ is always odd. $\square$