Solutions to Selected Exercises, HW #9

Assignment:

- T-BOP Section 3.2, page 73: Exercises 1, 2, 5, 6, 7, 8.
- T-BOP Section 3.3, page 77: Exercises 3, 4, 5, 6, 8, 11.
- T-BOP Section 3.4, page 84: Exercises 3, 4, 5, 7, 8, 10, 16.
- T-BOP Section 3.7, page 95: Exercises 3, 4ab, 10, 11, 12.

T-BOP, Section 3.2

Exercise 2. Airports are identified with 3-letter codes. For example, Richmond, Virginia has the code RIC, and Memphis, Tennessee has MEM. How many different 3-letter codes are possible?

Solution. $26^3 = 17{,}576$.

Exercise 6. You toss a coin, then roll a die, and then draw a card from a 52-card deck. How many different outcomes are there? How many outcomes are there in which the die lands on ⚂? How many outcomes are there in which the die lands on an odd number? How many outcomes are there in which the die lands on an odd number and the card is a King?

Solution. $2 \cdot 1 \cdot 52 = 104$ ways in which the die lands on ⚂. $2 \cdot 3 \cdot 52 = 312$ ways in which the die lands on an odd number. $2 \cdot 3 \cdot 4 = 24$ ways in which the die lands on an odd number and the card is a King.

Exercise 8. A coin is tossed 10 times in a row. How many possible sequences of heads and tails are there?

Solution. $2^{10} = 1024$ sequences.

T-BOP, Section 3.3

Exercise 4. Five cards are dealt off of a standard 52-card deck and lined up in a row. How many such lineups are there in which exactly one of the 5 cards is a queen?

Solution. $5 \cdot (4 \cdot 51 \cdot 50 \cdot 49 \cdot 48) = 93{,}398{,}400$.

Exercise 6. Consider lists made from the symbols A, B, C, D, E, with repetition allowed.

(a) How many such length-5 lists have at least one letter repeated?

Solution. The number with at least one letter repeated is the total number of length-5 lists minus the number with no letters repeated, which is

$$5^5 - 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3{,}005.$$

(b) How many such length-6 lists have at least one letter repeated?

Solution. There are only five letters, so any 6-letter list must have a repeat, so there are $5^6 = 15{,}625$ 6-letter lists (with at least one repeat).

Exercise 8. This problem concerns lists made from the letters $A, B, C, D, E, F, G, H, I, J$.

(a) How many length-5 lists can be made from these letters if repetition is not allowed and the list must begin with a vowel?

Solution. $3 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 9{,}072$.

(b) How many length-5 lists can be made from these letters if repetition is not allowed and the list must begin and end with a vowel?

Solution. $3 \cdot 8 \cdot 7 \cdot 6 \cdot 2 = 2{,}016$.

(c) How many length-5 lists can be made from these letters if repetition is not allowed and the list must contain exactly one A?

Solution. $5 \cdot (1 \cdot 9 \cdot 8 \cdot 7 \cdot 6) = 15{,}120$.

T-BOP, Section 3.4

Exercise 4. Using only pencil and paper, find the value of $\frac{100!}{95!}$.

Solution.

$$\frac{100!}{95!} = \frac{100 \cdot 99 \cdots 2 \cdot 1}{(95 \cdot 94 \cdots 2 \cdot 1)} = 100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 = 9{,}034{,}502{,}400.$$

Doing the multiplication at the end using only pencil and paper is hard. Pro tip: use a calculator for this step!

Exercise 8. Compute how many 7-digit numbers can be made from the digits $1, 2, 3, 4, 5, 6, 7$ if there is no repetition and the odd digits must appear in an unbroken sequence. (Examples: 3571264 or 2413576 or 2467531, etc., but not 7234615.)

Solution. There are 4 odd digits in this set of integers, so the number of unbroken sequences of these 4 odd digits is 4!. Such an unbroken sequence can occur as the first four digits in the 7-digit number, or as digits 2 through 5, or digits 3 though 6, or digits 4 through 7. For each of these four choices of where to put the odd digits, the number of ways that the remaining three spaces can be filled in with the remaining 3 even digits is $3! = 6$. So the total number of 7-digit numbers with no repetition and with the odd digits appearing in an unbroken sequence is

$$4! \cdot 4 \cdot 3! = 576.$$

Exercise 10. How many permutations of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are there in which the digits alternate even and odd? (For example, 2183470965.)

Solution. First, we form a permutation of the 5 even digits: there are 5! ways to do this. Then we form a permutation of the 5 odd digits: there are 5! ways to do this. Then we "zip" together the sequence of the evens with the sequence of the odds, so that they alternate. There are two ways to do this (either an even number comes first, or an odd number comes first). This gives

$$5! \cdot 5! \cdot 2 = 28{,}800$$

total permutations in which the digits alternate even and odd.

Exercise 16. How many 4-permutations are there of the set $\{A,B,C,D,E,F\}$ if whenever A appears in the permutation, it is followed by E?

Solution. Since any A must be followed by an E, an A can only appear in the first, second, or third place (it can't appear in the fourth, since then the E would have to be fifth, but there are only four places). Or an A might not appear at all.

The number of sequences starting with AE is $1 \cdot 1 \cdot 4 \cdot 3 = 12$, as is the number with AE in the second and third slots, as is the number with AE in the third and fourth slots. The number with no A is $5 \cdot 4 \cdot 3 \cdot 2$. So the total number where any A is followed by E is

$$12 + 12 + 12 + 5 \cdot 4 \cdot 3 \cdot 2 = 156.$$

T-BOP, Section 3.7

Exercise 4. This problem involves lists made from the letters T, H, E, O, R, Y, with repetition allowed.

(a) How many 4-letter lists are there that don't begin with T, or don't end in Y?

Solution. The total number of 4-letter lists is 6^4. The number that begin with T *and* end in Y is $1 \cdot 6 \cdot 6 \cdot 1 = 36$. So the number that don't begin with T *or* don't end in Y is

$$6^4 - 36 = 1{,}260.$$

(b) How many 4-letter lists are there in which the sequence of letters T, H, E appears consecutively (in that order)?

Solution. Such a list is either of the form $THE*$, where the "$*$" means "any one of the 6 letters," or of the form $*THE$. So there are $6 + 6 = 12$ such lists.

Exercise 10. How many 6-digit numbers are even or are divisible by 5?

Solution. An even number is one whose last digit is 0,2,4,6,8, and a number divisible by 5 is one whose last digit is either 0 or 5.

The number of 6-digit numbers that end in 0,2,4,6, or 8 is $9 \cdot 10^4 \cdot 5$. (There are 9 choices for the first digit, since the first digit can't be 0, then 10 choices for digits 2 through 5, then 5 choices for the last digit.). The number of 6-digit numbers that end in 0 or 5 is $9 \cdot 10^4 \cdot 2$. The number that are even *and* divisible by 5 are exactly those whose last digit is 0; There are $9 \cdot 10^4 \cdot 1$ of these. Sothe number that are even *or* are divisible by 5 is

$$9 \cdot 10^4 \cdot 5 + 9 \cdot 10^4 \cdot 2 - 9 \cdot 10^4 \cdot 1 = 540{,}000.$$

Here's another method: to say a 6-digit numbers is even or is divisible by 5 is to say it ends in 0,2,4,5,6, or 8. There are

$$9 \cdot 10^4 \cdot 6 = 540{,}000$$

such numbers.

Exercise 12. How many 5-digit numbers are there in which three of the digits are 7, or two of the digits are 2?

Solution. This is kinda hard, so bear with me.

First we count how many 5-digit numbers have exactly 3 7's. There are 10 choices for where to put the 3 7's (positions 1,2,3, or 1,2,4, or 1,2,5, or 1,3,4, or 1,3,5, or 1,4,5, or 2,3,4, or 2,3,5, or 2,4,5, or 3,4,5). For the first 6 of each of these choices (1,2,3, or 1,2,4, or 1,2,5, or 1,3,4, or 1,3,5, or 1,4,5), there are 9^2 ways of picking the other 3 digits. However, for the remaining 4 choices (2,3,4, or 2,3,5, or 2,4,5, or 3,4,5), there are only $8 \cdot 9$ ways, since the first digit can't be a 0.

So the number of 5-digit numbers with exactly 3 7's is

$$6 \cdot 9^2 + 4 \cdot 8 \cdot 9 = 774.$$

Next we count how many 5-digit numbers have exactly 2 2's. There are 10 choices for where to put the 2 2's (positions 1,2, or 1,3, or 1,4, or 1,5, or 2,3, or 2,4, or 2,5, or 3,4, or 3,5, or 4,5). For the first 4 of each of these choices (1,2, or 1,3, or 1,4, or 1,5), there are 9^3 ways of picking the other 3 digits. However, for the remaining 6 choices (2,3, or 2,4, or 2,5, or 3,4, or 3,5, or 4,5), there are only $8 \cdot 9^2$ ways, since the first digit can't be a 0.

So the number of 5-digit numbers with exactly 2 2's is

$$4 \cdot 9^3 + 4 \cdot 8 \cdot 9^2 = 6{,}804.$$

Now how many numbers have exactly 3 7's *and* 2 2's? Well again, there are 10 choices for where to put the 3 7's, after which there is only one choice for where to put the 2 2's. So there are 10 total numbers with exactly 3 7's *and* 2 2's.

So the number with 3 7's *or* two 2's is

$$774 + 6{,}804 - 10 = 7{,}568.$$