Solutions to Selected Exercises, HW #8

Assignment:

S-POP Part B(iv): Exercises B(iv) 1, 2, 3.

Exercise B(iv)-1. Show that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Solution. Let $\varepsilon > 0$. [Scratchwork: We want n large enough that

$$\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon.$$

But since $\sqrt{n}$ is positive for $n \in \mathbb{N}$, this condition is the same as $1/\sqrt{n} < \varepsilon$. Solving for n gives $\sqrt{n} > 1/\varepsilon$, or $n > 1/\varepsilon^2$. So this is what we write.]

Let $R = 1/\varepsilon^2$. Then $n > R \Rightarrow$

$$\left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{R}} = \frac{1}{\sqrt{1/\varepsilon^2}} = \frac{1}{1/\varepsilon} = \varepsilon.$$

So.

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0. \qquad \square$$

Exercise B(iv)-2. Show that

$$\lim_{n\to\infty} \frac{n^2 + (-1)^n\, n}{3n^2+1} = \frac{1}{3}.$$

Solution. Let $\varepsilon > 0$. [Scratchwork: We want n large enough that

$$\left|\frac{n^2 + (-1)^n\, n}{3n^2+1} - \frac{1}{3}\right| < \varepsilon.$$

But, getting a common denominator and using the triangle inequality, we have

$$\left|\frac{n^2 + (-1)^n\, n}{3n^2+1} - \frac{1}{3}\right| = \left|\frac{3(n^2 + (-1)^n\, n) - (3n^2+1)}{3(3n^2+1)}\right| = \left|\frac{3(-1)^n\, n - 1}{3(3n^2+1)}\right|$$
$$\leq \frac{|3(-1)^n\, n| + |-1|}{3(3n^2+1)} = \frac{3n+1}{3(3n^2+1)}.$$

By the hint, we have $3n+1 \leq 4n$, and $3n^2+1 > 3n^2$. So we get

$$\left|\frac{n^2 + (-1)^n\, n}{3n^2+1} - \frac{1}{3}\right| < \frac{4n}{3(3n^2)} = \frac{4}{9n}.$$

Solving $4/(9n) < \varepsilon$ for n is easy: we get $n > 4/(9\varepsilon)$. So this is what we write.] Let $R = 4/(9\varepsilon)$. Then $n > R \Rightarrow$

$$\begin{aligned}\left|\frac{n^2+(-1)^n n}{3n^2+1}-\frac{1}{3}\right| &= \left|\frac{3(n^2+(-1)^n n)-(3n^2+1)}{3(3n^2+1)}\right| = \left|\frac{3(-1)^n n-1}{3(3n^2+1)}\right| \\ &\le \frac{|3(-1)^n n|+|-1|}{3(3n^2+1)} = \frac{3n+1}{3(3n^2+1)} < \frac{4n}{3(3n^2)} = \frac{4}{9n} \\ &< \frac{4}{9R} = \frac{4}{9(4/(9\varepsilon))} = \frac{1}{1/\varepsilon} = \varepsilon.\end{aligned}$$

So

$$\lim_{n\to\infty}\frac{n^2+(-1)^n n}{3n^2+1} = \frac{1}{3}. \qquad \square$$

Exercise B(iv)-3. Show that

$$\lim_{n\to\infty}\frac{4n^3+n+\sin n}{7n^3+3} = \frac{4}{7}.$$

Solution. Let $\varepsilon > 0$. [Scratchwork: We want n large enough that

$$\left|\frac{4n^3+n+\sin n}{7n^3+3}-\frac{4}{7}\right| < \varepsilon.$$

But, getting a common denominator and using the triangle inequality, we have

$$\begin{aligned}\left|\frac{4n^3+n+\sin n}{7n^3+3}-\frac{4}{7}\right| &= \left|\frac{7(4n^3+n+\sin n)-4(7n^3+3)}{7(7n^3+3)}\right| = \left|\frac{7n+7\sin n-12}{7(7n^3+3)}\right| \\ &\le \frac{|7n|+|7\sin n|+|-12|}{7(7n^3+3)} \le \frac{7n+7+12}{7(7n^3+3)} = \frac{7n+19}{7(7n^3+3)}.\end{aligned}$$

(We used the fact that $|\sin n| \le 1$ always.) Now $19 \le 19n$, and $7n^3+3 > 7n^3$. So we get

$$\left|\frac{4n^3+n+\sin n}{7n^3+3}-\frac{4}{7}\right| < \frac{7n+19n}{7(7n^3)} = \frac{26n}{49n^3} = \frac{26}{49n^2}.$$

Solving $26/(49n^2) < \varepsilon$ for n gives us $1/n^2 < 49\varepsilon/26$, or $n^2 > 26/(49\varepsilon)$, or $n > \sqrt{26/(49\varepsilon)}$. So this is what we write.] Let $R = \sqrt{26/(49\varepsilon)}$. Then $n > R \Rightarrow$

$$\begin{aligned}\left|\frac{4n^3+n+\sin n}{7n^3+3}-\frac{4}{7}\right| &= \left|\frac{7(4n^3+n+\sin n)-4(7n^3+3)}{7(7n^3+3)}\right| = \left|\frac{7n+7\sin n-12}{7(7n^3+3)}\right| \\ &\le \frac{|7n|+|7\sin n|+|-12|}{7(7n^3+3)} \le \frac{7n+7+12}{7(7n^3+3)} = \frac{7n+19}{7(7n^3+3)} \\ &< \frac{7n+19n}{7(7n^3)} = \frac{26n}{49n^3} = \frac{26}{49n^2} < \frac{26}{49R^2} \\ &= \frac{26}{49\left(\sqrt{26/(49\varepsilon)}\right)^2} = \frac{26}{49\left(26/(49\varepsilon)\right)} = \frac{1}{1/\varepsilon} = \varepsilon.\end{aligned}$$

So

$$\lim_{n\to\infty}\frac{4n^3+n+\sin n}{7n^3+3} = \frac{4}{7}. \qquad \square$$