Solutions to Selected Exercises, HW #7

Assignment:

T-BOP Section 10.5, pages 195–196: Exercises 12, 15, 25, 26, 28.

Exercise 12. Prove that $9|(4^{3n}+8)$ for every integer $n \geq 0$.

Proof. Let $A(n)$ be the statement $9|(4^{3n}+8)$.

Step 1: Is $A(0)$ true?

$$4^{3\cdot 0}+8=9,$$

and $9|9$, so $A(0)$ is true.

Step 2: Assume $A(k):\ 9|(4^{3k}+8)$.

Then

$$\begin{aligned}4^{3(k+1)}+8 &= 4^{3k+3}+8\\ &= 4^{3k+3}+8+4^{3k}-4^{3k}\\ &= 4^{3k}+8+4^{3k+3}-4^{3k}\\ &= 4^{3k}+8+4^{3k}(4^3-1)\\ &= 4^{3k}+8+63\cdot 4^{3k}.\end{aligned}$$

Now $9|(4^{3k}+8)$ by the induction hypothesis, and $9|63$ since $63=9\cdot 7$. So, by Exercise B(i)-3 in S-POP, $9|(4^{3k}+8+63\cdot 4^{3k})$; that is, $9|4^{3(k+1)}+8$. So $A(k+1)$ follows.

So by the principle of mathematical induction, $A(n)$ is true for every integer $n \geq 0$. □

Exercise 28. Concerning the Fibonacci sequence, prove that

$$F_2+F_4+F_6+F_8+\cdots+F_{2n}=F_{2n+1}-1.$$

Proof. Let $A(n)$ be the given statement concerning Fibonacci numbers.

Step 1: Is $A(1)$ true?

$$\begin{aligned}F_{2\cdot 1} &\stackrel{?}{=} F_{2\cdot 1+1}-1\\ F_2 &\stackrel{?}{=} F_3-1\\ 1 &= 2-1\end{aligned}$$

so $A(1)$ is true.

Step 2: Assume

$$A(k):\ F_2+F_4+F_6+F_8+\cdots+F_{2k}=F_{2k+1}-1.$$

Then by the induction hypothesis and the recurrence relation $F_{n+2} = F_{n+1} + F_n$,

$$\begin{aligned}
F_2 + F_4 + F_6 + F_8 + \cdots + F_{2(k+1)} &= (F_2 + F_4 + F_6 + F_8 + \cdots + F_{2k}) + F_{2(k+1)} \\
&= F_{2k+1} - 1 + F_{2(k+1)} \\
&= F_{2k+2} + F_{2k+1} - 1 \\
&= F_{2k+3} - 1 \\
&= F_{2(k+1)+1} - 1,
\end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$