Solutions to Selected Exercises, HW #6

Assignment:

- S-POP Part B(v) : Exercises B(v) 1, 2, 3, 6.
- T-BOP Chapter 10 (page 195): Exercises 3, 4, 8.

S-POP, Part B(v)

Exercise 2. Use mathematical induction to prove that, for any positive integer n,

$$1 + 3 + 5 + 7 + \cdots + 2n - 1 = n^2.$$

That is, the sum of the first n *odd* positive integers is n^2.

Proof. Let $A(n)$ be the statement

$$1 + 3 + 5 + 7 + \cdots + 2n - 1 = n^2.$$

Step 1: Is $A(1)$ true?

$$1 = 1^2,$$

so $A(1)$ is true.

Step 2: Assume

$$A(k) : 1 + 3 + 5 + 7 + \cdots + 2k - 1 = k^2.$$

Then

$$\begin{aligned} 1 + 3 + 5 + 7 + \cdots + 2(k+1) - 1 &= 1 + 3 + 5 + 7 + \cdots + 2k - 1 + 2(k+1) - 1 \\ &= k^2 + 2(k+1) - 1 \\ &= k^2 + 2k + 1 \\ &= (k+1)^2, \end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

Exercise 3. Use mathematical induction to prove that, for any positive integer n,

$$\frac{d}{dx}x^n = nx^{n-1}$$

(pretend you didn't already know this, although it's OK to assume it's true for $n = 1$). Hint: for the inductive step, use the product rule.

Proof. Let $A(n)$ be the statement

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Step 1: Is $A(1)$ true?

$$\frac{d}{dx}x^1 = \frac{d}{dx}x = 1 = 1 \cdot x^{1-1},$$

so $A(1)$ is true.

Step 2: Assume

$$A(k) : \frac{d}{dx}x^k = kx^{k-1}.$$

Then by the product rule and the induction hypothesis,

$$\begin{aligned}\frac{d}{dx}x^{k+1} &= \frac{d}{dx}x^k \cdot x \\ &= \left(\frac{d}{dx}x^k\right)\cdot x + \left(\frac{d}{dx}x\right)\cdot x^k \\ &= \left(kx^{k-1}\right)\cdot x + \left(1\right)\cdot x^k \\ &= kx^k x + x^k = (k+1)x^k,\end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. □

Exercise 6. Let A_n be the statement

$$1 + 2 + 3 + \cdots + n = \frac{(2n+1)^2}{8}.$$

Prove that if $A(k)$ is true for any positive integer k, then so is $A(k+1)$. Is $A(n)$ true for all positive integers n? Explain your answer.

Solution. First we prove that $A(k) \Rightarrow A(k+1)$. So assume

$$A(k) : 1 + 2 + 3 + \cdots + k = \frac{(2k+1)^2}{8}.$$

Then

$$\begin{aligned}1 + 2 + 3 + \cdots + (k+1) &= (1 + 2 + 3 + \cdots + k) + k + 1 \\ &= \frac{(2k+1)^2}{8} + k + 1 = \frac{(2k+1)^2}{8} + \frac{8(k+1)}{8} \\ &= \frac{(2k+1)^2 + 8(k+1)}{8} = \frac{4k^2 + 4k + 1 + 8k + 8}{8} = \frac{4k^2 + 12k + 9}{8} \\ &= \frac{(2k+3)^2}{8} = \frac{(2(k+1)+1)^2}{8},\end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

However, $A(n)$ is *not* true for all $n \in \mathbb{N}$. For example, $A(1)$ is the statement 1=9/8, which is false.

The point of this exercise is that it's not enough just to show that $A(k) \Rightarrow A(k+1)$. You also need to prove the base case $A(1)$. Without that, the statement $A(n)$ might not even be true for a single positive integer n.

T-BOP Chapter 10 (page 195)

Prove the following statements with either induction, strong induction, or proof by smallest counterexample.

Exercise 4. If $n \in N$, then

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Proof. Let $A(n)$ be the statement of the proposition.

Step 1: Is $A(1)$ true?

$$\begin{aligned} 1 \cdot 2 &\stackrel{?}{=} \frac{1(1+1)(1+2)}{3} \\ 2 &= 2. \end{aligned}$$

so $A(1)$ is true.

Step 2: Assume

$$A(k) : 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}.$$

Then

$$\begin{aligned} &1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + (k+1)(k+1+1) \\ &= \big(1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + k(k+1)\big) + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3} = \frac{(k+1)((k+1)+1)((k+1)+2)}{3}, \end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. □

Exercise 8. If $n \in N$, then

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

Proof. Let $A(n)$ be the statement of the proposition.

Step 1: Is $A(1)$ true?

$$\frac{1}{2!} \stackrel{?}{=} \frac{1}{(1+1)!}$$
$$\frac{1}{2} = \frac{1}{2}.$$

so $A(1)$ is true.

Step 2: Assume

$$A(k) : \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}.$$

Then

$$\begin{aligned}
&\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k+1}{((k+1)+1)!} \\
&= \left(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!}\right) + \frac{k+1}{((k+1)+1)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{k+2}{(k+2)!} + \frac{k+1}{(k+2)!} \\
&= 1 + \frac{-(k+2)+k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!},
\end{aligned}$$

so $A(k+1)$ follows. (To get a common denominator, we have used the fact $1/(k+1)! = (k+2)/(k+2)!$.) Therefore, $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$