Solutions to Selected Exercises, HW #5

Assignment:

- T-BOP Section 2.7, page 55: Exercises 1–5, 9, 10.
- Supply a negation, in quantifier form, of each of the Exercises 3–5, 9, and 10 in T-BOP, Section 2.7. Then state whether your negated statement is true or false.
- S-POP Part B(iii), pages 8–9: Exercises B(iii)-1 through B(iii)-5.

T-BOP, Section 2.7

Write the following as English sentences. Say whether they are true or false.

Exercise 2. $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}, x^n \geq 0$: For any real number x, there is a natural number n such that $x^n \geq 0$. True: no matter what x is, take $n = 2$; $x^2 \geq 0$ always.

Exercise 4. $\forall x \in \mathscr{P}(\mathbb{N}), X \subseteq \mathbb{R}$: Every subset of $\mathbb{N}$ is a subset of $\mathbb{R}$. True: every set of integers is also a set of real numbers.

Exercise 10. $\exists m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m = n + 5$: There is some integer $m = n + 5$ for every integer n. False: were there such an m, we would have, for example, $m = 4+5$ and $m = 27 + 5$, so $9 = 32$, which is false.

Negations of Exercises 3–5, 9, and 10 in T-BOP, Section 2.7.

Exercise 4. The negation of the statement

$$\forall x \in \mathscr{P}(\mathbb{N}), X \subseteq \mathbb{R}$$

is the statement

$$\exists x \in \mathscr{P}(\mathbb{N}), X \not\subseteq \mathbb{R}.$$

This negation is false, since the original statement is true.

Exercise 10. The negation of the statement

$$\exists m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m = n + 5$$

is the statement

$$\forall m \in \mathbb{Z}, \exists n \in \mathbb{Z}, m \neq n + 5.$$

This negation is true, since the original statement is false. Alternatively note that, given $m \in \mathbb{Z}$, let $n = m + 46$: then $n + 5 = m + 51$, and $m \neq m + 51$, so $m \neq n + 5$.

S-POP Part B(iii)

Exercise B(iii)-1.

Theorem. $\forall m \in \mathbb{Z}, 6|m(m+1)(m+2)$.

Proof. Let $m \in \mathbb{Z}$. It is enough to show that $m(m+1)(m+2)$ is both even and divisible by 3, since, by Exercise B(i)-9 in S-POP, this will imply that $m(m+1)(m+2)$ is divisible by 6.

1. First we show that $m(m+1)(m+2)$ is even, as follows. We know that m is either even or odd. We consider two subcases:

   (a) If m is even, then $m = 2k$ for some $k \in \mathbb{Z}$, so

   $$m(m+1)(m+2) = (2k)(2k+1)(2k+2) = 2\big(k(2k+1)(2k+2)\big),$$

   so $m(m+1)(m+2)$ is even.

   (b) If m is odd, then $m = 2k+1$ for some $k \in \mathbb{Z}$, so

   $$m(m+1)(m+2) = (2k+1)(2k+2)(2k+3) = 2\big((2k+1)(k+1)(2k+3)\big),$$

   so $m(m+1)(m+2)$ is even.

   In either case, $m(m+1)(m+2)$ is even.

2. Next, we show that $m(m+1)(m+2)$ is divisible by 3. By the division algorithm, we can write $m = 3\ell + r$, where r equals 0, 1, or 2. We consider three subcases:

   (a) If $r = 0$, then $m = 2\ell$ for some $\ell \in \mathbb{Z}$, so

   $$m(m+1)(m+2) = (3\ell)(3\ell+1)(3\ell+2) = 3\big(\ell(3\ell+1)(3\ell+2)\big),$$

   so $m(m+1)(m+2)$ is divisible by 3.

   (b) If $r = 1$, then $m = 3\ell + 1$ for some $\ell \in \mathbb{Z}$, so

   $$m(m+1)(m+2) = (3\ell+1)(3\ell+2)(3\ell+3) = 3\big((3\ell+1)(3\ell+2)(\ell+1)\big),$$

   so $m(m+1)(m+2)$ is divisible by 3.

   (c) If $r = 2$, then $m = 3\ell + 2$ for some $\ell \in \mathbb{Z}$, so

   $$m(m+1)(m+2) = (3\ell+2)(3\ell+3)(3\ell+4) = 3\big((3\ell+2)(\ell+1)(3\ell+4)\big),$$

   so $m(m+1)(m+2)$ is divisible by 3.

   In all cases, $m(m+1)(m+2)$ is divisible by 3.

Since $m(m+1)(m+2)$ is both even and divisible by 3, it is divisible by 6, as required. □

Exercise B(iii)-2.

Theorem. $\forall x, y \in \mathbb{R}$,

$$x^2 + y^2 \geq 6x + 4y - 15.$$

Proof. Let $x, y \in \mathbb{R}$. We wish to show that $x^2 + y^2 \geq 6x + 4y - 15$, which is equivalent to showing that

$$x^2 + y^2 - 6x - 4y + 15 \geq 0.$$

But note that $x^2 - 6x = (x-3)^2 - 9$ and $x^2 - 4y = (x-2)^2 - 4$. So what we need to show is that

$$(x-3)^2 - 9 + (x-2)^2 - 4 + 15 \geq 0,$$

or in other words, that

$$(x-3)^2 + (x-2)^2 + 2 \geq 0.$$

But $(x-3)^2$ is the square of a real number, so it's always ≥ 0. Similarly, so is $(x-2)^2$. So

$$(x-3)^2 + (x-2)^2 + 2 \geq 0 + 0 + 2 \geq 0,$$

as required. □

Exercise B(iii)-3.

Theorem. $\exists p \in \{\text{prime numbers}\}, p > 100$.

Proof. Let $p = 101$. Then p is prime, as can be shown by simply checking whether any number n, between 2 and 100, divides p.

So $\exists p \in \{\text{prime numbers}\}, p > 100$. □

Exercise B(iii)-4.

Theorem. $\exists k \in \mathbb{Z}$ such that k can be expressed as a sum of two squares in two different ways.

Proof. Let $k = 50$. Then $k = 1^2 + 7^2 = 5^2 + 5^2$. So k can be expressed as a sum of squares in two different ways. □

Exercise B(iii)-5.

(a) **Theorem.** $\forall x \in \mathbb{R}, \exists y \in \mathbb{R},\ x > y$.

Proof. Assume $x \in \mathbb{R}$. Let $y = x - 1$. Then $x > y$.

Therefore, $\forall x \in \mathbb{R}, \exists y \in \mathbb{R},\ x > y$. □

(b) **Theorem.** $\sim (\exists y \in \mathbb{R}, \forall x \in \mathbb{R},\ x > y)$.

Proof. The statement

$$\sim (\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x > y)$$

is equivalent to the statement

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x \leq y.$$

So assume $y \in \mathbb{R}$. Let $x = y - 1$. Then $x \leq y$.

So

$$\sim (\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x > y).$$

$\square$