Solutions to Selected Exercises, HW #3

Assignment:

- S-POP, Section B(i): Exercises B(i)-4 through B(i)-10.
- S-POP, Section B(ii): Exercises B(ii)-5 through B(ii)-7.
- T-BOP Chapter 8 (page 171): Exercises 14, 16.

S-POP, Part B(i)

Exercise B(i)-5. Supply a direct proof of Proposition B(i)-2_E.

SOLUTION:

Theorem. If m^2 is an odd number, then m is an odd number.

Proof. Let m^2 be an odd number; write $m^2 = 2\ell + 1$, where $\ell \in \mathbb{Z}$. Now every number is either even or odd, so we can write $m = 2k + r$, where $k \in \mathbb{Z}$ and either $r = 0$ or $r = 1$. But then

$$m^2 = (2k + r)^2 = 4k^2 + 4kr + r^2,$$

where r is either 0 or 1. Setting our two expressions for m^2 equal, we get

$$4k^2 + 4kr + r^2 = 2\ell + 1,$$

or, solving for r^2,

$$r^2 = 2\ell + 1 - (4k^2 + 4kr) = 2(\ell - 2k^2 - 2kr) + 1 = 2n + 1,$$

where $n \in \mathbb{Z}$. So r^2 is odd, so r can't equal 0 (because $0^2 = 0$ is even), so $r = 1$, so $m = 2k + 1$, so m is odd. □

Exercise B(i)-6. Using contraposition prove that, if n is not divisible by 4, then n is not divisible by 12.

SOLUTION:

Proof. Suppose n is divisible by 12. Then $n = 12k$ for some $k \in \mathbb{Z}$. But then, since $12 = 4 \cdot 3$, we have $n = (4 \cdot 3) \cdot k = 4 \cdot (3k) = 4m$, where $m \in \mathbb{Z}$. So n is divisible by 4.

So if n is not divisible by 4, then n is not divisible by 12. □

Exercise B(i)-8. Consider the converse to the statement of Exercise B(i)-3(a). Is this converse statement true? If so, prove it. If not, show that it's false by counterexample.

SOLUTION:

The converse is FALSE. For example, let $a = 7, b = 10$, and $c = 11$. Then $a|(b + c)$ (since $7|21$), but $a \nmid b$ and $a \nmid c$.

Exercise B(i)-9. Use the method outlined in Proposition B(i)-3 to show that an integer n is divisible by 6 if, and only if, n is both even and divisible by 3.

SOLUTION:

Proof. Suppose n is divisible by 6. Then $n = 6k$ where $k \in \mathbb{Z}$. Since $6 = 2 \cdot 3 = 3 \cdot 2$, we therefore have $n = (2 \cdot 3) \cdot k = 2 \cdot (3k) = 3\ell$ where $\ell \in \mathbb{Z}$, and $n = (3 \cdot 2) \cdot k = 3 \cdot (2k) = 3m$ where $m \in \mathbb{Z}$. So n is even and divisible by 3.

Next, suppose n is even and is divisible by 3. Since n is divisible by 3, we have $n = 3k$ for some $k \in \mathbb{Z}$. But then, since n is even, k must be even, otherwise n would be the product of two odd numbers (3 and k), and would therefore be odd, by Exercise B(i)-1(b). So $k = 2m$ for some $m \in \mathbb{Z}$, so $n = 3(2m) = 6m$, so n is divisible by 6.

So n is divisible by 6 if, and only if, n is both even and divisible by 3. $\square$

S-POP, Part B(ii)

Exercise B(ii)-5. Using the strategy of Proposition B(ii)-2, prove that

$$\mathbb{Z} = 3\mathbb{Z} \cup 3\mathbb{Z} + 1 \cup 3\mathbb{Z} + 2.$$

SOLUTION: (Note: we used notation like $1 + 3\mathbb{Z}$ in class, but of course, this is the same as $3\mathbb{Z} + 1$).

1) First, we show that $\mathbb{Z} \subseteq 3\mathbb{Z} \cup 3\mathbb{Z} + 1 \cup 3\mathbb{Z} + 2$: let $m \in \mathbb{Z}$. By the division algorithm, we have $m = 3q + r$ where $q, r \in \mathbb{Z}$ and either $r = 0$, $r = 1$, or $r = 2$. In the first case we have $m = 3q$, so $m \in 3\mathbb{Z}$. In the second case we have $m = 3q + 1$, so $m \in 3\mathbb{Z} + 1$. In the third case we have $m = 3q + 2$, so $m \in 3\mathbb{Z} + 2$. So either $m \in 3\mathbb{Z}$ or $m \in 3\mathbb{Z} + 1$ or $m \in 3\mathbb{Z} + 2$, so by definition of union, $m \in 3\mathbb{Z} \cup 3\mathbb{Z} + 1 \cup 3\mathbb{Z} + 2$.

So $\mathbb{Z} \subseteq 3\mathbb{Z} \cup 3\mathbb{Z} + 1 \cup 3\mathbb{Z} + 2$.

2) Next, we show that $3\mathbb{Z} \cup 3\mathbb{Z} + 1 \cup 3\mathbb{Z} + 2 \subseteq \mathbb{Z}$: let $m \in 3\mathbb{Z} \cup 3\mathbb{Z} + 1 \cup 3\mathbb{Z} + 2$. Then $m \in 3\mathbb{Z}$ or $m \in 3\mathbb{Z} + 1$ or $m \in 3\mathbb{Z} + 2$. In each case, $m \in \mathbb{Z}$, since each of these three sets is a set of integers. So $3\mathbb{Z} \cup 3\mathbb{Z} + 1 \cup 3\mathbb{Z} + 2 \subseteq \mathbb{Z}$.

Therefore, $\mathbb{Z} = 3\mathbb{Z} \cup 3\mathbb{Z} + 1 \cup 3\mathbb{Z} + 2$. $\square$

T-BOP, Chapter 8

Exercise 16.

Theorem. If A, B, and C are sets, then $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Proof. Suppose A, B, and C are sets.

1) Let $x \in A \times (B \cup C)$. Then by definition of Cartesian product, $x = (a, d)$, where $a \in A$ and $d \in B \cup C$. By definition of union, $d \in B$ or $d \in C$. In the first case, since

$a \in A$, we have $x \in A \times B$, so certainly $x \in A \times B$ or $x \in A \times C$, so by definition of union, $x \in (A \times B) \cup (A \times C)$. In the second case, since $a \in A$, we have $x \in A \times C$, so certainly $x \in A \times B$ or $x \in A \times C$, so by definition of union, $x \in (A \times B) \cup (A \times C)$. So in either case, $x \in (A \times B) \cup (A \times C)$.

So $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

2) Next, let $x \in (A \times B) \cup (A \times C)$. Then by definition of union, $x \in A \times B$ or $x \in A \times C$. In the first case, $x = (a, b)$, where $a \in A$ and $b \in B$. But then certainly $b \in B$ or $b \in C$, so $b \in B \cup C$ by definition of union, so $x \in A \times (B \cup C)$. In the second case, $x = (a, c)$, where $a \in A$ and $c \in C$. But then certainly $c \in B$ or $c \in C$, so $c \in B \cup C$ by definition of union, so $x \in A \times (B \cup C)$. So in either case, $x \in A \times (B \cup C)$.

So $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Therefore, $A \times (B \cup C) = (A \times B) \cup (A \times C)$. $\square$