Solutions to Selected Exercises, HW #1

Assignment:

- S-POP, Part B(i): Exercises B(i)-1, 2, 3.
- T-BOP, Section 1.1 (pages 7–8): Exercises A(1, 3, 6, 9, 14), B(17,18, 26, 27), C(29, 32, 34, 38).
- T-BOP, Chapter 4 (pages 126–127): 5, 7, 14.

S-POP, Part B(i)

Exercise B(i)-1:

(a) Prove that the sum of two odd numbers is even.

SOLUTION:

Theorem. The sum of two odd numbers is even.

Proof. We may state this in $P \Rightarrow Q$ form as follows: if $a, b \in \mathbb{Z}$ are odd, then $a + b$ is even.

So assume $a, b \in \mathbb{Z}$. We may write $a = 2k + 1$ and $b = 2\ell + 1$, where $k, \ell \in \mathbb{Z}$. But then

$$a + b = 2k + 1 + 2\ell + 1 = 2k + 2\ell + 2 = 2(k + \ell + 1) = 2m,$$

where $m = k + \ell + 1 \in \mathbb{Z}$. So, by definition of even integer, $a + b$ is even.

So the sum of two odd numbers is even. □

(b) Prove that the product of two odd numbers is odd.

SOLUTION:

Theorem. The product of two odd numbers is odd.

Proof. We may state this in $P \Rightarrow Q$ form as follows: if $a, b \in \mathbb{Z}$ are odd, then ab is odd.

So assume $a, b \in \mathbb{Z}$. We may write $a = 2k + 1$ and $b = 2\ell + 1$, where $k, \ell \in \mathbb{Z}$. But then

$$ab = (2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1 = 2m + 1,$$

where $m = 2k\ell + k + \ell \in \mathbb{Z}$. So, by definition of odd integer, ab is odd.

So the product of two odd numbers is odd. □

Exercise B(i)-3. Let a, b, and c be integers. Recall that we say "a divides b," written $a \big| b$, if there exists an integer q such that $b = aq$.

(a) Prove that, if $a \big| b$ and $a \big| c$, then $a \big| (b+c)$.

SOLUTION:

Theorem. If $a, b, c \in \mathbb{Z}$, $a \big| b$, and $a \big| c$, then $a \big| (b+c)$.

Proof. Assume that $a, b, c \in \mathbb{Z}$, and that $a \big| b$, and $a \big| c$. We may then write $b = am$ and $c = an$, where $m, n \in \mathbb{Z}$. But then

$$b + c = am + an = a(m+n) = a\ell,$$

where $\ell = m + n \in \mathbb{Z}$. So, by definition of divisibility, $a \big| (b+c)$.

So $a, b, c \in \mathbb{Z}$, $a \big| b$, and $a \big| c \Rightarrow a \big| (b+c)$. $\square$

(b) Prove that, if $a \big| b$, then $a \big| nb$ for any integer n.

SOLUTION:

Theorem. If $a, b \in \mathbb{Z}$ and $a \big| b$, then $a \big| nb$ for any integer n.

Proof. Assume $a, b \in \mathbb{Z}$ and $a \big| b$. We may then write $b = am$, where $m \in \mathbb{Z}$. But then, if n is an integer,

$$nb = nam = a(nm).$$

So, by definition of divisibility, $a \big| nb$.

So $a, b \in \mathbb{Z}$ and $a \big| b \Rightarrow a \big| nb$ for any integer n. $\square$

T-BOP, Section 1.1

Exercises A:

1. $\{5x - 1 : x \in \mathbb{Z}\} = \{\ldots, -16, -11, -6, -1, 4, 9, 14, \ldots\}$.

3. $\{x \in \mathbb{Z} : -2 \leq x < 7\} = \{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$.

6. $\{x \in \mathbb{R} : x^2 = 9\} = \{-3, 3\}$.

9. $\{x \in \mathbb{R} : \sin \pi x = 0\} = \mathbb{Z}$.

14. $\{5x : x \in \mathbb{Z}, |2x| \leq 8\} = \{-20, -15, -10, -5, 0, 5, 10, 15, 20\}$.

Exercises C:

29. $\big|\{\{1\}, \{2, \{3, 4\}, \emptyset\}\}\big| = 3$.

32. $\big|\{\{\{1, 4\}, a, b, \{\{3, 4\}\}, \{\emptyset\}\}\}\big| = 1$.

34. $\big|\{x \in \mathbb{N} : |x| < 10\}\big| = 9$.

38. $\big|\{x \in \mathbb{N} : 5x \leq 20\}\big| = 4$.

T-BOP, Chapter 4

Exercise 7: Suppose $a, b \in \mathbb{Z}$. If $a|b$, then $a^2|b^2$.

SOLUTION:

Theorem. If $a, b \in \mathbb{Z}$ and $a\big|b$, then $a^2\big|b^2$.

Proof. Assume $a, b \in \mathbb{Z}$ and $a\big|b$. We may then write $b = am$, where $m \in \mathbb{Z}$. But then

$$b^2 = (am)^2 = a^2m^2 = a^2n,$$

where $n = m^2$. So, by definition of divisibility, $a^2\big|b^2$.

So $a, b \in \mathbb{Z}$ and $a\big|b \Rightarrow a^2\big|b^2$. $\square$

Exercise 14: If $n \in \mathbb{Z}$, then $5n^2 + 3n + 7$ is odd.

SOLUTION:

Theorem. If $n \in \mathbb{Z}$, then $5n^2 + 3n + 7$ is odd.

Proof. Assume $n \in \mathbb{Z}$. We consider two cases:

(a) n is even. Then we can write $n = 2k$ where $k \in \mathbb{Z}$. We then have

$$\begin{aligned} 5n^2 + 3n + 7 &= 5(2k)^2 + 3(2k) + 7 \\ &= 5 \cdot 4k^2 + 6k + 7 \\ &= 20k^2 + 6k + 7 \\ &= 2(10k^2 + 3k + 3) + 1 \\ &= 2m + 1, \end{aligned}$$

where $m = 10k^2 + 3k + 3 \in \mathbb{Z}$. So, by definition of odd integer, $5n^2 + 3n + 7$ is odd.

(b) n is odd. Then we can write $n = 2k + 1$ where $k \in \mathbb{Z}$. We then have

$$\begin{aligned} 5n^2 + 3n + 7 &= 5(2k + 1)^2 + 3(2k + 1) + 7 \\ &= 5 \cdot (4k^2 + 4k + 1) + 6k + 3 + 7 \\ &= 20k^2 + 20k + 5 + 6k + 3 + 7 \\ &= 20k^2 + 26k + 15 \\ &= 2(10k^2 + 13k + 7) + 1 \\ &= 2\ell + 1, \end{aligned}$$

where $\ell = 10k^2 + 13k + 7 \in \mathbb{Z}$. So, by definition of odd integer, $5n^2 + 3n + 7$ is odd.

Since n must be either even or odd, we see that, in all cases, $5n^2 + 3n + 7$ is odd. $\square$