Some notes on random variables: probability mass functions, expected value, and the binomial distribution

1 Random variables, probability mass functions, and expected value

Section 1 Exercises.

Exercise 1.1. Find the probability mass function and the expected value for the random variables Y and Q of Example 1.3 above.

Solution. Write an outcome as a string of two digits from 1 to 6, where the first digit is the number appearing on the first die and the second digit is the number on the second. Let S be the sample space of 36 outcomes.

First, let Y be the sum of the numbers on the two dice. We compute:

$$P(Y=2)=\frac{|\{11\}|}{|S|}=\frac{1}{36}, \qquad P(Y=3)=\frac{|\{12,21\}|}{|S|}=\frac{2}{36},$$
$$P(Y=4)=\frac{|\{13,31,22\}|}{|S|}=\frac{3}{36}, \qquad P(Y=5)=\frac{|\{14,41,23,32\}|}{|S|}=\frac{4}{36},$$
$$P(Y=6)=\frac{|\{15,51,24,42,33\}|}{|S|}=\frac{5}{36}, \qquad P(Y=7)=\frac{|\{16,61,25,52,34,43\}|}{|S|}=\frac{6}{36},$$
$$P(Y=8)=\frac{|\{26,62,35,53,44\}|}{|S|}=\frac{5}{36}, \qquad P(Y=9)=\frac{|\{36,63,45,54\}|}{|S|}=\frac{4}{36},$$
$$P(Y=10)=\frac{|\{46,64,55\}|}{|S|}=\frac{3}{36}, \qquad P(Y=11)=\frac{|\{56,65\}|}{|S|}=\frac{2}{36},$$
$$P(Y=12)=\frac{|\{66\}|}{|S|}=\frac{1}{36}.$$

Note that the probabilities add up to $36/36=1$.

Also,

$$E(Y)=2\cdot\frac{1}{36}+3\cdot\frac{2}{36}+4\cdot\frac{3}{36}+5\cdot\frac{4}{36}+\cdots+11\cdot\frac{2}{36}+12\cdot\frac{1}{36}=\frac{252}{36}=7.$$

Next, let Q be the smallest of the numbers on the two dice. We compute:

$$P(Q=1)=\frac{|\{11,12,21,13,31,14,41,15,51,16,61\}|}{|S|}=\frac{11}{36}, \qquad P(Q=4)=\frac{|\{44,45,54,46,64\}|}{|S|}=\frac{5}{36},$$
$$P(Q=2)=\frac{|\{22,23,32,24,42,25,52,26,62\}|}{|S|}=\frac{9}{36}, \qquad P(Q=5)=\frac{|\{55,56,65\}|}{|S|}=\frac{3}{36},$$
$$P(Q=3)=\frac{|\{33,34,43,35,53,36,63\}|}{|S|}=\frac{7}{36}, \qquad P(Q=6)=\frac{|\{66\}|}{|S|}=\frac{1}{36}.$$

Note that the probabilities add up to $36/36=1$.

Also,

$$E(Q) = 1 \cdot \frac{11}{36} + 2 \cdot \frac{9}{36} + 3 \cdot \frac{7}{36} + 4 \cdot \frac{5}{36} + 5 \cdot \frac{3}{36} + 6 \cdot \frac{1}{36} = \frac{252}{36} = \frac{91}{36} = 2.52778.$$

Exercise 1.2. Flip four fair coins, and record each outcome as a string of H's and T's, as in Example 1.2 above, except that here we have only four coins, not six.

(a) Write down the sample space S for this experiment. That is, write down, explicitly, all possible outcomes.

Solution.

$$S = \{HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, THHT, \\ THTH, TTHH, HTTH, HTTT, THTT, TTHT, TTTH, TTTT\}.$$

(b) Let X be the number of heads showing on your flip of the four coins. Find the probability mass function for X, as well as the expected value $E(X)$. Use the method of Example **??** above. That is: for each value x of X, *count* how many outcomes have x heads, and divide by the total number of possible outcomes to find $P(X = x)$.

Why does your answer for $E(X)$ make intuitive sense?

Solution. We compute:

$$\begin{aligned}
P(X = 0) &= \frac{|\{TTTT\}|}{|S|} = \frac{1}{16} = 0.0625, \\
P(X = 1) &= \frac{|\{HTTT, THTT, TTHT, TTTH\}|}{|S|} = \frac{4}{16} = 0.25, \\
P(X = 2) &= \frac{|\{HHTT, HTHT, THHT, THTH, TTHH, HTTH\}|}{|S|} = \frac{6}{16} = 0.375, \\
P(X = 3) &= \frac{|\{HHHT, HHTH, HTHH, THH\}|}{|S|} = \frac{4}{16} = 0.25, \\
P(X = 4) &= \frac{|\{TTTT\}|}{|S|} = \frac{1}{16} = 0.0625.
\end{aligned}$$

Note that the probabilities add up to $16/16 = 1$.

Also,

$$E(X) = 0 \cdot \frac{1}{16} + 1 \cdot \frac{4}{16} + 2 \cdot \frac{6}{16} + 3 \cdot \frac{4}{16} + 4 \cdot \frac{1}{16} = \frac{32}{16} = 2.$$

The answer makes intuitive sense because there are 4 fair coins, so on average we would expect 2 of them to come up heads.

2 The binomial distribution

Section 2 Exercises.

Exercise 2.1. A 40% three-point field goal shooter attempts five three-point shots in a game.

(a) Find the probability mass function for the number X of three-point shots made out of the five. Confirm that your probabilities add up to one.

Solution. We compute:

$$P(X=0) = \binom{5}{0} \times .4^0 \times .6^5 = 0.07776,$$
$$P(X=1) = \binom{5}{1} \times .4^1 \times .6^4 = 0.2592,$$
$$P(X=2) = \binom{5}{2} \times .4^2 \times .6^3 = 0.3456,$$
$$P(X=3) = \binom{5}{3} \times .4^3 \times .6^2 = 0.2304,$$
$$P(X=4) = \binom{5}{4} \times .4^4 \times .6^1 = 0.0768,$$
$$P(X=5) = \binom{5}{5} \times .4^5 \times .6^0 = 0.01024.$$

Note that the probabilities add up to

$$0.07776 + 0.2592 + 0.3456 + 0.2304 + 0.0768 + 0.01024 = 1.$$

(b) Find the probability that the player makes at least two three-point shots, out of the five taken. Hint: it might be easier to first compute the probability that they make fewer than two.

Solution.

$$\begin{aligned} P(\text{at least two shots made}) &= 1 - P(\text{fewer than two made}) \\ &= 1 - P(X=0) - P(X=1) = 1 - 0.07776 - 0.2592 = 0.66304. \end{aligned}$$

(c) Find the expected number of three-point shots made. Use the method of Example 2.4 above (that is, use the definition of expected value, and the probabilities that you computed in part (a) of this exercise).

Solution.

We compute that

$$E(X) = 0 \cdot 0.07776 + 1 \cdot 0.2592 + 2 \cdot 0.3456 + 3 \cdot 0.2304 + 4 \cdot 0.0768 + 5 \cdot 0.01024 = 2.$$

(d) Why does your answer to part (c) of this exercise make intuitive sense?

Solution. We would expect to shooter to make 40% of the 5 shots, and $0.4 \cdot 5 = 2$.

Exercise 2.2. Consider the following game. You pay \$10, and pick a number from 1 through 6. A fair die is rolled three times. You are awarded \$0 if your number does not come up at all in the three rolls; you are awarded \$20 if your number comes up once, \$30 if it comes up twice, and \$40 if it comes up all three times.

Let X be the number of times your chosen number comes up. Then X is the number of successes in three trials of a binomial experiment, with $P(\text{success}) = 1/6$.

(a) Find the probability mass function for X.

Solution. We compute:

$$\begin{aligned}
P(X=0) &= \binom{3}{0} \times \left(\frac{1}{6}\right)^0 \times \left(\frac{5}{6}\right)^3 = 0.5787, \\
P(X=1) &= \binom{3}{1} \times \left(\frac{1}{6}\right)^1 \times \left(\frac{5}{6}\right)^2 = 0.3472, \\
P(X=2) &= \binom{3}{0} \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^1 = 0.0694, \\
P(X=3) &= \binom{3}{3} \times \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^0 = 0.0046.
\end{aligned}$$

Note that the probabilities add up to

$$0.5787 + 0.3472 + 0.0694 + 0.0046 = 0.999$$

(there is some roundoff error).

(b) Should you play the game? Hint: your expected payoff, in dollars, is

$$-10 \cdot P(X=0) + 10 \cdot P(X=1) + 20 \cdot P(X=2) + 30 \cdot P(X=3).$$

Solution. The expected payoff is

$$-10 \cdot 0.5787 + 10 \cdot 0.3472 + 20 \cdot 0.0694 + 30 \cdot 0.0046 = 0.999 = -0.789$$

dollars. You should not play, since the expected payoff is negative.

Exercise 2.3. Find the probability mass function and the expected value for the random variable X of Example 1.2 (the example where you flip six fair coins). Use Theorem 2.3 above (that is, *do not* actually count outcomes, as you did in Exercise 1.2 above).

Solution. We compute:

$$
\begin{aligned}
P(X=0) &= \binom{6}{0} \times .5^0 \times .5^6 = 0.015625, \\
P(X=1) &= \binom{6}{1} \times .5^1 \times .6^5 = 0.09375, \\
P(X=2) &= \binom{6}{2} \times .5^2 \times .5^4 = 0.234375, \\
P(X=3) &= \binom{6}{3} \times .5^3 \times .5^3 = 0.3125, \\
P(X=4) &= \binom{6}{4} \times .5^4 \times .5^2 = 0.234375, \\
P(X=5) &= \binom{6}{5} \times .5^5 \times .5^1 = 0.09375, \\
P(X=6) &= \binom{6}{6} \times .5^6 \times .5^0 = 0.015625.
\end{aligned}
$$

Note that the probabilities add up to 1.

Also,

$$
\begin{aligned}
E(X) &= 0 \cdot 0.015625 + 1 \cdot 0.09375 + 2 \cdot 0.234375 + 3 \cdot 0.3125 + 4 \cdot 0.234375 + 5 \cdot 0.09375 + 6 \cdot 0.015625 \\
&= 3.
\end{aligned}
$$