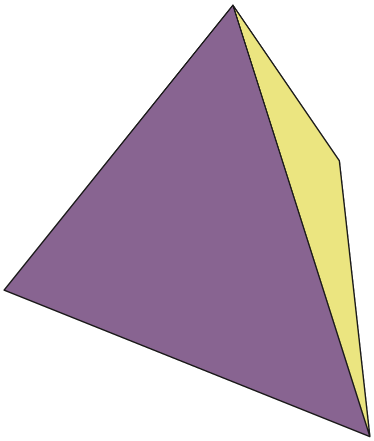
A *Platonic solid* is a polyhedron that is *regular* (every face of the polyhedron is a regular polygon, and all faces are congruent, meaning they have the same size and shape), and has *regular vertices* (this means the same number of edges meet at each vertex). A Platonic solid is said to *have symbol* $\{p, q\}$ if each of its polygonal faces has p edges, and if q edges meet at each vertex. For example, the tetrahedron:



is a Platonic solid of symbol $\{3, 3\}$.

We are now going to prove (by filling in the blanks):

Theorem. There are exactly five Platonic solids.

Proof. let P be a polyhedron with symbol $\{p, q\}$. Let F denote the number of faces of P. By what we've assumed about the symbol of P, every face of P has $\underline{\quad p \quad}$ edges. So, since there are F faces, in total we would have $\underline{\quad pF \quad}$ edges, except for the fact that each edge is shared by *two* faces; so in fact, we only have half as many edges. That is, we only have $\underline{\quad pF/2 \quad}$ edges. In other words, if E denotes the number of edges of P, then F can be expressed in terms of E as follows:

$$F = \underline{\quad 2E/p \quad}.$$

Now, let V denote the number of *vertices* of P. By what we've assumed about the symbol of P, $\underline{\quad q \quad}$ edges of P meet at each vertex. So, since there are V vertices, in total we would have $\underline{\quad qV \quad}$ edges, except for the fact that each edge emanates from *two* vertices; so in fact, we only have half as many edges. That is, we only have $\underline{\quad qV/2 \quad}$ edges. In other words, if E denotes the number of edges of P, then V can be expressed in terms of E as follows:

$$V = \underline{\quad 2E/q \quad}.$$

But we also know that, for any polyhedron P, $V - E + F = 2$ always. This is called *Euler's formula*; we did this in class. Into Euler's formula, let's plug the above formula for F in terms of p and E, and for V in terms of q and E; we get

$$\underline{\quad 2E/q - E + 2E/p \quad} = 2.$$

Now divide the above formula by $2E$ on both sides, and then add $1/2$ to both sides of the result, to get

$$\frac{1}{q} + \frac{1}{p} = \frac{1}{E} + \underline{\quad \frac{1}{2} \quad}. \tag{1}$$

Since E is positive, so is $1/E$; so the above tells us that

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{2}. \tag{2}$$

What possible values of p and q can make this work? Well first of all, we find that either p or q must be less than 4, because if not then both p and q would be greater than or equal to $\underline{\quad 4 \quad}$, so that both $1/p$ and $1/q$ would be less than or equal to $\underline{\quad 1/4 \quad}$, so that

$$\frac{1}{p} + \frac{1}{q} \leq \underline{\quad \frac{1}{2} \quad},$$

contradicting equation **(2)**.

So we see that either $p \leq \underline{\quad 3 \quad}$ (an integer) or $q \leq \underline{\quad 3 \quad}$ (an integer). On the other hand, since a polygon has at least three sides, we must have $p \geq \underline{\quad 3 \quad}$; also, since at least three edges must emanate from each vertex of a polyhedron (think about it!), we have $q \geq \underline{\quad 3 \quad}$ as well.

To summarize the previous paragraph: it must be the case that *either* p *or* q is less than or equal to $\underline{\quad 3 \quad}$, AND that *both* p *and* q are greater than or equal to $\underline{\quad 3 \quad}$. This can only happen if either p or q is *exactly equal* to $\underline{\quad 3 \quad}$, and the other is greater than or equal to $\underline{\quad 3 \quad}$.

Suppose the first of these situations holds: that is, suppose $p = \underline{\quad 3 \quad}$ and $q \geq \underline{\quad 3 \quad}$. Putting $p = \underline{\quad 3 \quad}$ into equation **(1)**, and solving for $1/q$, gives us gives us

$$\frac{1}{q} = \frac{1}{E} + \underline{\quad \frac{1}{6} \quad}. \tag{3}$$

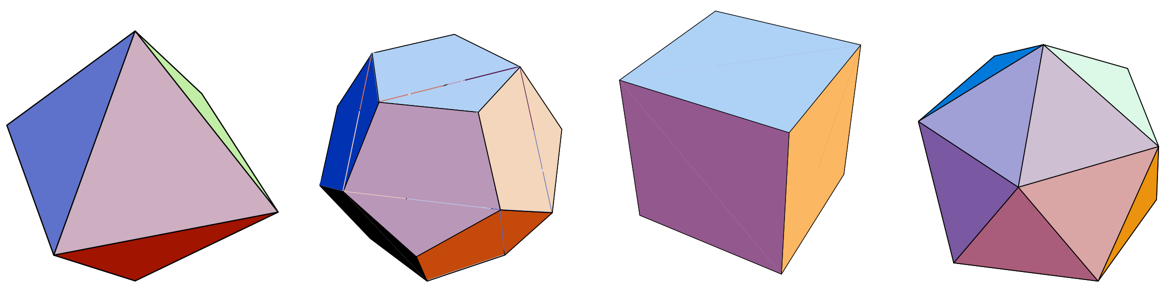
(fill in the blank with the correct fraction). Since $1/E$ is positive, this implies $1/q > \underline{\quad 1/6 \quad}$, or $q < \underline{\quad 6 \quad}$ (an integer). But since q is an integer and we're assuming $q \geq 3$, we have $q = \underline{\quad 3 \quad}$, $q = \underline{\quad 4 \quad}$, or $q = \underline{\quad 5 \quad}$. To summarize: the case $p = \underline{\quad 3 \quad}$ and $q \geq \underline{\quad 3 \quad}$ gives us three possibilities for the symbol of P: $\{p, q\} = \underline{\quad \{3, 3\} \quad}$, $\{p, q\} = \underline{\quad \{3, 4\} \quad}$, or $\{p, q\} = \underline{\quad \{3, 5\} \quad}$.

Now let's consider the other possibility, namely, when $q = \underline{\quad 3 \quad}$ and $p \geq \underline{\quad 3 \quad}$. Putting $q = \underline{\quad 3 \quad}$ into equation **(1)** tells us that

$$\frac{1}{p} = \frac{1}{E} + \underline{\quad \frac{1}{6} \quad} \tag{4}$$

(fill in the blank with the correct fraction). Since $1/E$ is positive, this implies $1/p > \underline{\quad 1/6 \quad}$, or $p < \underline{\quad 6 \quad}$ (an integer). But since p is an integer and we're assuming $p \geq 3$, we have $p = \underline{\quad 3 \quad}$, $p = \underline{\quad 4 \quad}$, or $p = \underline{\quad 5 \quad}$. To summarize: the case $q = \underline{\quad 3 \quad}$ and $p \geq \underline{\quad 3 \quad}$ gives us three possibilities for the symbol of P: $\{p, q\} = \underline{\quad \{3,3\} \quad}$, $\{p, q\} = \underline{\quad \{4,3\} \quad}$, or $\{p, q\} = \underline{\quad \{5,3\} \quad}$.

So, in total, we have the following *five* distinct possibilities for the symbol $\{p, q\}$ of our polyhedron P: $\{p, q\} = \underline{\quad \{3,3\} \quad}$, $\underline{\quad \{3,4\} \quad}$, $\underline{\quad \{3,5\} \quad}$, $\underline{\quad \{4,3\} \quad}$, or $\underline{\quad \{5,3\} \quad}$. To show that we actually *do* have five distinct Platonic solids, we need to show that each of these possibilities really does correspond to a solid. We already saw that the $\underline{\quad \text{tetrahedron} \quad}$ has symbol $\underline{\quad \{3,3\} \quad}$. To finish we consider the following picture, depicting the octahedron, dodecahedron, cube, and icosahedron respectively:



From this picture, we see that the the octahedron has symbol $\underline{\quad \{3,4\} \quad}$, the dodecahedron has symbol $\underline{\quad \{5,3\} \quad}$, the cube has symbol $\underline{\quad \{4,3\} \quad}$, and the icosahedron has symbol $\underline{\quad \{3,5\} \quad}$. So all five symbols *are* actually realized, and the proof is O.V.A.H.