Take-Home Final Exam, due December 7

You are to complete this midterm on your own, without assistance from any human or other resources, except for: lectures notes from this class, T-BOP, and S-POP, yourself, and me (you are free to ask me questions, some of which I might not answer completely for you). You can also use paper and something to write with.

Complete this exam on your own paper. (If it's easier for you, you may fill out the truth tables for exercise 3 directly on page 2 of this exam, and hand in that page for your solution to exercise 3.) Be *neat*. **You must complete, and attach to your exam, a copy the cover sheet at the end of this exam.**

Your exam must be turned in at the *beginning* of class on Monday, December 7. Late exams will not be accepted.

Good luck!

1. Consider the recurrence relation

$$a_1 = 1, \quad a_2 = 5, \quad a_n = 5a_{n-1} - 6a_{n-2} \quad (n \geq 3).$$

(a) Compute a_3, a_4, and a_5. 19, 65, 211.

(b) Use the method of generating functions to solve the above recurrence relation. Hint: $1 - 5x + 6x^2 = (1-3x)(1-2x)$.

We define

$$A(x) = \sum_{n=1}^{\infty} a_n x^n.$$

Then

$$\begin{aligned} A(x) &= a_1 x^1 + a_2 x^2 + \sum_{n=3}^{\infty} a_n x^n \\ &= x + 5x^2 + \sum_{n=3}^{\infty} (5a_{n-1} - 6a_{n-2}) x^n \\ &= x + 5x^2 + 5\sum_{n=3}^{\infty} a_{n-1} x^n - 6\sum_{n=3}^{\infty} a_{n-2} x^n. \end{aligned}$$

Into the first sum on the right, we substitute $m = n - 1$; into the second, we substitute

$m = n - 2$. We get

$$\begin{aligned} A(x) &= x + 5x^2 + 5\sum_{m=2}^{\infty} a_m x^{m+1} - 6\sum_{m=1}^{\infty} a_m x^{m+2} \\ &= x + 5x^2 + 5x\sum_{m=2}^{\infty} a_m x^m - 6x^2\sum_{m=1}^{\infty} a_m x^m \\ &= x + 5x^2 + 5x\left(\sum_{m=1}^{\infty} a_m x^m - a_1 x^1\right) - 6x^2\sum_{m=1}^{\infty} a_m x^m \\ &= x + 5x^2 + 5x\big(A(x) - x\big) - 6x^2 A(x) \\ &= x + (5x - 6x^2)A(x). \end{aligned}$$

Solving for $A(x)$ gives

$$A(x) = \frac{x}{1 - 5x + 6x^2}. \tag{1}$$

Since $1 - 5x + 6x^2 = (1-3x)(1-2x)$, we write **(1)** as

$$A(x) = \frac{A}{1-3x} + \frac{B}{1-2x}. \tag{2}$$

Getting a common denominator on the right gives

$$A(x) = \frac{A(1-2x) + B(1-3x)}{1-5x+6x^2} = \frac{(A+B) - (2A+3B)x}{1-5x+6x^2}. \tag{3}$$

Equating numerators of **(1)** and **(3)** gives

$$x = (A+B) - (2A+3B)x$$

or, matching up coefficients of like powers of x,

$$0 = A + B \text{ and } 1 = -2A - 3B.$$

Solving for A and B gives $A = 1$ and $B = -1$, or, by **(2)**,

$$A(x) = \frac{1}{1-3x} - \frac{1}{1-2x}. \tag{4}$$

Now, applying the geometric series expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

to **(4)**, we get

$$A(x) = \sum_{n=0}^{\infty}(3x)^n - \sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty}[3^n - 2^n]x^n. \qquad \textbf{(5)}$$

Matching up coefficients of x^n, for $n \geq 1$, on the left and right sides of **(5)** gives

$$a_n = 3^n - 2^n,$$

and we're done.

(c) Check your work by using your answer to part (b) to compute a_1, a_2, a_3, a_4, and a_5, and making sure your answers agree with results obtained above. $a_1 = 3-2 = 1$; $a_2 = 3^2-2^2 = 5$; $a_3 = 3^3-2^3 = 27-8 = 19$; $a^4 = 3^4-2^4 = 81-16 = 65$; $a_5 = 3^5-2^5 = 243-32 = 211$.

2.

Show that, for $k, \ell \in \mathbb{N}$ and $\ell \leq k$,

$$\binom{k+1}{\ell} = \binom{k}{\ell} + \binom{k}{\ell-1}.$$

Hint: you don't need induction or anything like that. This is just a computation, using the definition

$$\binom{k}{\ell} = \frac{k!}{\ell!(k-\ell)!}.$$

Apply this definition to the terms on the right-hand side of the equality you're trying to prove; do some algebra involving common denominators etc., and show that you get the left-hand side.

$$\binom{k}{\ell} + \binom{k}{\ell-1} = \frac{k!}{\ell!(k-\ell)!} + \frac{k!}{(\ell-1)!(k-\ell+1)!}.$$

Now note that $\ell! = \ell(\ell-1)!$ and $(k-\ell+1)! = (k-\ell+1)(k-\ell)!$. So, if we multiply the numerator and denominator of the first term on the right by $k-\ell+1$, and multiply the numerator and denominator of the second term on the right by ℓ, we get

$$\binom{k}{\ell} + \binom{k}{\ell-1} = \frac{k!(k-\ell+1)}{\ell!(k-\ell+1)!} + \frac{k!\ell}{\ell!(k-\ell+1)!} = \frac{k!((k-\ell+1)+\ell)}{\ell!(k-\ell+1)!}$$

$$= \frac{k!(k+1)}{\ell!(k-\ell+1)!} = \frac{(k+1)!}{\ell!(k+1-\ell)!} = \binom{k+1}{\ell}.$$

3. Given statements A and B, we define $A|B$, pronounced "A nand B," by

$$A|B = \sim(A \wedge B).$$

(The symbol "|" is sometimes called the "Sheffer stroke.") In other words, $A|B$ is defined by this truth table:

A	B	$A\|B$
T	T	F
T	F	T
F	T	T
F	F	T

(a) Use the truth table below to show that

$$A|A = \sim A.$$

A	$\sim A$	$A\|A$
T	F	F
F	T	T

(b) Use the truth table below to show that

$$(A|A)|(B|B) = A \vee B.$$

A	B	$A\|A$	$B\|B$	$(A\|A)\|(B\|B)$	$A \vee B$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

(c) Use the truth table below to show that

$$(A|B)|(A|B) = A \wedge B.$$

A	B	$A\|B$	$(A\|B)\|(A\|B)$	$A \wedge B$
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	T	F	F

4. Imagine flipping 10 fair coins. ("Fair" means that the coin landing heads is just as likely as the coin landing tails. We'll assume that heads and tails are the only possible outcomes. That is, the coin can't land on its edge, hover in midair forever, etc.)

(a) How many possible outcomes are there? (Think of an outcome as being a list, of length 10, of H's and T's (e.g. HTTHHTTTHT), where the first letter in the list designates how the first coin landed, and so on). $2^{10} = 1024$.

(b) How many outcomes have the first and third coin landing heads? Please express your answer as a single natural number (and show your work to indicate where that number came from). $2^8 = 256$.

(c) How many outcomes have the first *or* the third coin, or both, landing heads? Please express your answer as a single natural number (and show your work to indicate where that number came from). $2^9 + 2^9 - 2^8 = 768$.

(d) How many outcomes have exactly two coins landing heads? Please express your answer as a single natural number (and show your work to indicate where that number came from). $\binom{10}{2} = 45$.

(e) Let's call an outcome *roughly symmetric* if the number of heads differs from the number of tails by at most two. What's more likely: an outcome that's roughly symmetric, or one that's not? Please explain. Roughly symmetric. To say that the number of heads differs from the number of tails by at least two is to say there are 4, 5, or 6 heads. The number of such outcomes is

$$\binom{10}{4} + \binom{10}{5} + \binom{10}{6} = 672.$$

So the number of outcomes that are not roughly symmetric is

$$1024 - 672 = 352.$$

Since $672 > 352$, more outcomes are roughly symmetric.

5. Let $A = \{a, b, c, d, e, f\}$.

(a) How many different possible relations are there on A? Hint: recall that a relation is a subset of $A \times A$. 2^{36} =68,719,476,736.

(b) Consider the relation R on A defined by

$$R = \{(a, a), (a, e), (c, a), (d, b)\}.$$

Make R into an equivalence relation R' by adding to R as few elements of $A \times A$ as possible. That is, find a set R' such that $R \subseteq R'$, R' is an equivalence relation on A, and $|R'|$ is as small as possible.

$R' = \{(a,a),(a,e),(e,a),(e,e),(c,a),(a,c),(c,c),(e,c),(c,e),(d,b),(b,d),(b,b),(d,d),(f,f)\}$.

(c) For the relation R' that you found above, define the *equivalence class* $[\alpha]$ of any $\alpha \in A$ in the usual way:

$$[\alpha] = \{\beta \in A \colon \beta R' \alpha.\}.$$

Write down the equivalence class $[\alpha]$ of each of the six elements $\alpha \in A$. Then, *partition* A into equivalence classes. That is, find equivalence classes that are disjoint, and whose union is all of A. A is the disjoint union of $[a]$, $[b]$, and $[f]$.

6. Let $a, b \in \mathbb{Z}$ and $m \in \mathbb{N}$.

(a) Use mathematical induction to prove that, for any $n \in \mathbb{N}$,

$$a \equiv b \pmod{m} \Rightarrow a^n \equiv b^n \pmod{m}.$$

Hint for the inductive step:
$a^{k+1} - b^{k+1} = a(a^k - b^k) + b^k(a - b)$.

Proof. Let A_n denote the claim we're trying to prove, for $n \in \mathbb{N}$.

Base step: does $a \equiv b \pmod{m} \Rightarrow a^1 \equiv b^1 \pmod{m}$? Yes, clearly. So A_1 is true.

Inductive step: assume

$$A_k : \quad a \equiv b \pmod{m} \Rightarrow a^k \equiv b^k \pmod{m}.$$

From this we want to deduce

$$A_{k+1} : \quad a \equiv b \pmod{m} \Rightarrow a^{k+1} \equiv b^{k+1} \pmod{m}.$$

So assume $a \equiv b \pmod{m}$. Them $m|(a-b)$ and moreover, by A_k, $m|(a^k - b^k)$. But it follows that

$$m|(a(a^k - b^k) + b^k(a - b));$$

that is, $m|(a^{k+1} - b^{k+1})$. So $a^{k+1} \equiv b^{k+1} \pmod{m}$. Thus, A_{k+1} is true.

So A_1 is true, and $A_k \Rightarrow A_{k+1}$. So, by mathematical induction, A_n is true for all $n \in \mathbb{N}$.

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(b) Find four distinct integers a, all between 0 and 11, such that $a^2 \equiv 1 \pmod{12}$. $a = 1, 5, 7, 11$.

(c) Is the converse to part (a) of this exercise true? Please explain. No. $1^2 \equiv 5^2 (\text{mod } 12)$, but $1 \not\equiv 5 (\text{mod } 12)$.

(d) For integers a and b, and for $n \in \mathbb{N}$, it's true that, if $a^n = b^n$, then $a = \pm b$. Is the analogous statement true mod m (for $m \in \mathbb{N}$)? That is: is it true that, if $a^n \equiv b^n \pmod{m}$, then $a \equiv \pm b \pmod{m}$? Please explain. No. $1^2 \equiv 5^2 (\text{mod } 12)$, but $1 \not\equiv \pm 5 (\text{mod } 12)$.