**DIY notes on recurrence relations and generating functions (SOLUTIONS)**

Suppose we have a sequence

$$B_1, B_2, B_3 \dots$$

that satisfies some recurrence relation, expressing $B_n$ in terms of previous $B_m$'s: say

$$B_n = c_0 + c_1 B_{n-1} + c_2 B_{n-2} + \dots + c_k B_{n-k}.$$

(To keep things simple, we'll assume that $c_0, c_1, \dots c_k$ are constants, although in principle, they might depend on $n$.) Suppose we also have *initial conditions,* meaning the first few (in this case, the first $k$) $B_m$'s are known.

**Example:** Suppose

$$Q_n = 4Q_{n-1} + 12Q_{n-2} \text{ for } n \geq 3;\ Q_1 = 1;\ Q_2 = 4.$$

**Exercise 1:** Write down (as integers) the first five $Q_n$'s (including the first two, as specified above).

1, 4, 28, 160, 976

The method of *generating functions* can often be applied to deduce a closed (non-recursive) formula for $B_n$, in the following way.

**Step 1.** Define the *generating function*

$$B(x) = \sum_{n=1}^{\infty} B_n x^n$$

for the $B_n$'s.

**Exercise 2:** Define a generating function $Q(x)$ for the above sequence of $Q_n$'s.

$$Q(x) = \sum_{n=1}^{\infty} Q_n x^n$$

**Step 2.** Use the recurrence relation and initial conditions to find a *simple* expression for $B(x)$.

There are three tricks that are often useful here:

- First, split off terms from your series $B(x)$, so that the recurrence relation applies to each of the remaining terms in the sum.
- After applying the recurrence relation to the resulting sum, make changes in your indices of summation, to obtain sums that have only $B_m$'s (and not $B_{m-1}$'s, $B_{m-2}$'s, etc.) in them.
- Adjust the resulting infinite sums (by adding and subtracting the appropriate terms) so that they start at the same index at which the sum $B(x)$ starts (in our case, at 1).

Having completed these steps, you will, with luck, get an equation you can solve for $B(x)$, thus completing Step 2.

**Exercise 3:** Show that

$$Q(x) = \frac{x}{1 - 4x - 12x^2}.$$

I'll get you started, with the first trick:

$$Q(x) = \sum_{n=1}^{\infty} Q_n x^n = Q_1 x^1 + Q_2 x^2 + \sum_{n=3}^{\infty} Q_n x^n.$$

Hints for proceeding: plug in for $Q_1$, $Q_2$, and $Q_n$ on the right (for $Q_n$, use the recurrence relation). Break up your resulting sum into two sums. In the first of these sums, put $m = n - 1$; in the second, put $m = n - 2$. (This is the second trick.) Then use the third trick as necessary so that both sums on the right start at $m = 1$. Your result should be an equation that can be solved for $Q(x)$.

OK go ahead; here's the start, again:

$$\begin{aligned} Q(x) &= \sum_{n=1}^{\infty} Q_n x^n = Q_1 x^1 + Q_2 x^2 + \sum_{n=3}^{\infty} Q_n x^n \\ &= x + 4x^2 + \sum_{n=3}^{\infty} (4Q_{n-1} + 12Q_{n-2}) x^n \\ &= x + 4x^2 + 4\sum_{n=3}^{\infty} Q_{n-1} x^n + 12 \sum_{n=3}^{\infty} Q_{n-2} x^n. \end{aligned}$$

Substitute $m = n - 1$ in the first series on the right, and $m = n - 2$ in the second, to get

$$\begin{aligned} Q(x) &= x + 4x^2 + 4\sum_{m=2}^{\infty} Q_m x^{m+1} + 12 \sum_{m=1}^{\infty} Q_m x^{m+2} \\ &= x + 4x^2 + 4\left(\sum_{m=1}^{\infty} Q_m x^{m+1} - Q_1 x^2\right) + 12 \sum_{m=1}^{\infty} Q_m x^{m+2}. \end{aligned}$$

$$= x + 4x^2 + 4\left(x \sum_{m=1}^{\infty} Q_m x^m - x^2\right) + 12x^2 \sum_{m=1}^{\infty} Q_m x^m$$
$$= x + 4x^2 + 4xQ(x) - 4x^2 + 12x^2 Q(x)$$
$$= x + Q(x)(4x + 12x^2).$$

Solving for $Q(x)$ gives

$$Q(x) = \frac{x}{1 - 4x - 12x^2}.$$

**Step 3.** Expand your *simple* expression for $B(x)$ into a power series. Typically, this will entail a partial fraction decomposition of your expression for $B(x)$.

**Exercise 4.** Find constants $U$ and $V$ such that

$$\frac{x}{1 - 4x - 12x^2} = \frac{U}{1 - 6x} + \frac{V}{1 + 2x}.$$

Hint: get a common denominator on the right. Show that this denominator is the same as the denominator on the left. So you can equate numerators, and match up powers of $x$ in these numerators, to solve for $U$ and $V$.

The equation

$$\frac{x}{1 - 4x - 12x^2} = \frac{U}{1 - 6x} + \frac{V}{1 + 2x}$$

gives, upon getting a common denominator on the right,

$$\frac{x}{1 - 4x - 12x^2} = \frac{U(1 + 2x) + V(1 - 6x)}{(1 - 6x)(1 + 2x)} = \frac{U + V + x(2U - 6V)}{1 - 4x - 12x^2}.$$

Equating coefficients of like powers of $x$ in the numerators, on the left and right sides above, gives $U + V = 0$ and $2U - 6V = 1$. These equations are easily solved to give $U = 1/8$ and $V = -1/8$. So

$$\frac{x}{1 - 4x - 12x^2} = \frac{1}{8}\left(\frac{1}{1 - 6x} - \frac{1}{1 + 2x}\right).$$

**Exercise 5.** Use the geometric series formula

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,$$

together with your answers from exercises 3 and 4 above, to express $Q(x)$ as an explicit power series in $x$.

$$Q(x) = \frac{x}{1-4x-12x^2} = \frac{1}{8}\left(\frac{1}{1-6x} - \frac{1}{1+2x}\right) = \frac{1}{8}\left(\sum_{n=0}^{\infty}(6x)^n - \sum_{n=0}^{\infty}(-2x)^n\right)$$
$$= \frac{1}{8}\sum_{n=0}^{\infty}(6^n - (-2)^n)x^n.$$

**Step 4.** Match coefficients of like powers of $x$ in your original series for $B(x)$ (from Step 1) with those in your new series (from Step 3), to obtain a formula for $B_n$.

**Exercise 6.** Combine the results of exercises 2 and 5 above to find an explicit, closed formula for $Q_n$, for $n \geq 1$.

$$Q_n = \frac{1}{8}(6^n - (-2)^n).$$

**Exercise 7.** Plug $n = 1, 2, 3, 4, 5$ directly into your formula from exercise 6, to verify your results from exercise 1.

$$Q_1 = \frac{1}{8}(6 - (-2)) = 1.$$

$$Q_2 = \frac{1}{8}(6^2 - (-2)^2) = 4.$$

$$Q_3 = \frac{1}{8}(6^3 - (-2)^3) = 28.$$

$$Q_4 = \frac{1}{8}(6^4 - (-2)^4) = 160.$$

$$Q_5 = \frac{1}{8}(6^5 - (-2)^5) = 976.$$

Here is one more worked:

**Example.** Use the method of generating functions to solve the recurrence relation

$$C_1 = 3, \quad C_2 = 9, \quad C_n = 3C_{n-1} + 10C_{n-2} \quad (n \geq 3).$$

**Solution.** We define

$$C(x) = \sum_{n=1}^{\infty} C_n x^n.$$

Then

$$\begin{aligned} C(x) &= C_1 x^1 + C_2 x^2 + \sum_{n=3}^{\infty} C_n x^n \\ &= 3x + 9x^2 + \sum_{n=3}^{\infty} (3C_{n-1} + 10C_{n-2}) x^n \\ &= 3x + 9x^2 + 3\sum_{n=3}^{\infty} C_{n-1} x^n + 10\sum_{n=3}^{\infty} C_{n-2} x^n. \end{aligned}$$

Into the first sum on the right, we substitute $m = n - 1$; into the second, we substitute $m = n - 2$. We get

$$\begin{aligned} C(x) &= 3x + 9x^2 + 3\sum_{m=2}^{\infty} C_m x^{m+1} + 10\sum_{m=1}^{\infty} C_m x^{m+2} \\ &= 3x + 9x^2 + 3x\sum_{m=2}^{\infty} C_m x^m + 10x^2\sum_{m=1}^{\infty} C_m x^m \\ &= 3x + 9x^2 + 3x\left(\sum_{m=1}^{\infty} C_m x^m - C_1 x^1\right) + 10x^2\sum_{m=1}^{\infty} C_m x^m \\ &= 3x + 9x^2 + 3x\big(C(x) - 3x\big) + 10x^2 C(x) \\ &= 3x + (3x + 10x^2)C(x). \end{aligned}$$

Solving for $C(x)$ gives

$$C(x) = \frac{3x}{1 - 3x - 10x^2}. \tag{1}$$

Since $1 - 3x - 10x^2 = (1 - 5x)(1 + 2x)$, we write **(1)** as

$$C(x) = \frac{A}{1 - 5x} + \frac{B}{1 + 2x}. \tag{2}$$

Getting a common denominator on the right gives

$$C(x) = \frac{A(1+2x) + B(1-5x)}{1-3x-10x^2} = \frac{(A+B) + (2A-5B)x}{1-3x-10x^2}. \qquad \textbf{(3)}$$

Equating numerators of **(1)** and **(3)** gives

$$3x = (A+B) + (2A-5B)x$$

or, matching up coefficients of like powers of $x$,

$$0 = (A+B) \text{ and } 3 = 2A - 5B.$$

Solving for $A$ and $B$ gives $A = 3/7$ and $B = -3/7$, or, by **(2)**,

$$C(x) = \frac{3}{7}\left[\frac{1}{1-5x} - \frac{1}{1+2x}\right]. \qquad \textbf{(4)}$$

Now, applying the geometric series expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

to **(4)**, we get

$$C(x) = \frac{3}{7}\left[\sum_{n=0}^{\infty}(5x)^n - \sum_{n=0}^{\infty}(-2x)^n\right] = \frac{3}{7}\sum_{n=0}^{\infty}\big[5^n - (-2)^n\big]x^n. \qquad \textbf{(5)}$$

Matching up coefficients of $x^n$, for $n \geq 1$, on the left and right sides of **(5)** gives

$$C_n = \frac{3}{7}\big[5^n - (-2)^n\big],$$

and we're done.