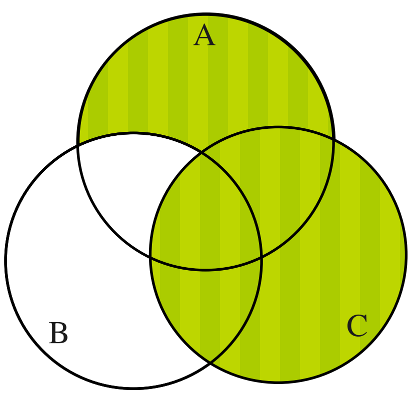
Take-Home Midterm Exam, due October 12 (SOLUTIONS)

1. Supply your own list of *at least five* shoulds and shouldnt's of mathematical communication. See S-POP, Part B, for more details. It's OK if one or more items on your list is something we have discussed in class. But be thoughtful. That is, your list should reflect things that are relevant to your own writing and thinking about mathematics.

- When communicating mathematics, you *should* write in complete sentences.
- When communicating mathematics, you *should* use connecting words, like "therefore," "also," and so on, between statements. This helps make the meaning clear.
- When communicating mathematics, you *should* use parentheses to indicate groupings – of numbers, sets, etc. For example, $A \cup B - C$ is ambiguous: does it mean $A \cup (B - C)$, or $(A \cup B) - C$?
- When communicating mathematics, you *should* always remember to state your assumptions. For example, to prove "If n is divisible by 4, then n is even," you *shouldn't* start your proof with "We write $n = 4k$ for some $k \in \mathbb{Z}$." You *should*, instead, start with something like "Assume n is divisible by 4. Then we can write $n = 4k$ for some $k \in \mathbb{Z}$."
- When communicating mathematics, you *shouldn't* ignore the above four "shoulds."

2: Exercises from T-BOP: Complete the following exercises from the text.

(a) Section 1.7 (p. 23): 10.



12. $(A - B) \cup (B \cap C)$.

(b) Section 1.8 (p. 28): 13. First question: Yes. Proof: let $x \in \cup_{\alpha \in J} A_\alpha$. Then, by definition of union, $x \in A_\beta$ for some $\beta \in J$. But, since $J \subseteq I$, it follows that $\beta \in I$, so $A_\beta \subseteq \cup_{\alpha \in I} A_\alpha$, so $x \in \cup_{\alpha \in I} A_\alpha$. So $\cup_{\alpha \in J} A_\alpha \subseteq \cup_{\alpha \in I} A_\alpha$. (Roughly speaking: the *more* sets you union, the *larger* a set you tend to get.)

Second question: no. For example, let $J = \{1, 2\}$ and $I = \{1, 2, 3\}$. Then certainly $J \subseteq I$. Now let $A_1 = \{a, b, c\}$, $A_2 = \{b, c, d, e\}$, and $A_3 = \{c, d, e, f\}$. Then

$$\cap_{\alpha \in J} A_\alpha = A_1 \cap A_2 = \{b, c\} \text{ and } \cap_{\alpha \in I} A_\alpha = A_1 \cap A_2 \cap A_3 = \{c\},$$

so $\cap_{\alpha \in J} A_\alpha$ is *not* a subset of $\cap_{\alpha \in I} A_\alpha$. (Roughly speaking: the *more* sets you intersect, the *smaller* a set you tend to get.)

14. Yes. Proof: let $x \in \cap_{\alpha \in I} A_\alpha$. Then, by definition of intersection, $x \in A_\alpha$ for every $\alpha \in I$. But, since $J \subseteq I$, it follows that $x \in A_\alpha$ for every $\alpha \in J$. So $x \in \cap_{\alpha \in J} A_\alpha$. So $\cap_{\alpha \in I} A_\alpha \subseteq \cap_{\alpha \in J} A_\alpha$.

3: Quantifiers. For this exercise, you might want to recall that the negation of a statement like "$\exists x \in X : Q(x)$" is the statement "$\forall x \in X, \sim Q(x)$." Here, $\sim Q(x)$ denotes the negation of $Q(x)$; that is, $\sim Q(x)$ means "not $Q(x)$." (In other words, $\sim Q(x)$ means "$Q(x)$ is false.") Similarly, the negation of a statement like "$\forall x \in X : Q(x)$" is the statement "$\exists x \in X, \sim Q(x)$."

(a) Let $Q(x, y)$ be a statement regarding objects x and y (in some universe U). How would you express the negation of the statement $\forall x \in X, \exists y \in Y : Q(x, y)$ in terms of $\sim Q(x, y)$? $\exists x \in X : \forall y \in Y, \sim Q(x, y)$

(b) Express the negation of the statement $\exists x \in X : \forall y \in Y, Q(x, y)$ in terms of $\sim Q(x, y)$.

$\forall x \in X, \exists y \in Y : \sim Q(x, y)$

(c) Express the negation of the statement $\forall x \in X, \exists y \in Y : \forall z \in Z, Q(x, y, z)$ in terms of $\sim Q(x, y, z)$. (Here, $Q(x, y, z)$ is some statement involving objects x, y, z.)

$\exists x \in X : \forall y \in Y, \exists z \in Z : \sim Q(x, y)$

4: More quantifiers. Identify each of the following statements as true or false (circle "**T**" or "**F**"). If a statement is true, explain why (you don't need to provide a complete proof; just a sentence or two will do). If a statement is false, provide a counterexample to the statement.

(a) $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z} : (m - n) | k$. **T** **F**

Given any integers m and n, let $k = 0$ (or $k = m - n$). Then $(m - n) | k$.

(b) $\exists k \in \mathbb{Z} : \forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, (m - n) | k$. **T** **F**

Let $k = 0$. Then, given any integers m and n, we have $(m - n) | k$.

(c) $\sim\big(\forall m \in \mathbb{Z}, \exists k \in \mathbb{Z} : \forall n \in \mathbb{Z}, (m - n) | k\big)$. **T** **F**

Consider the statement $\forall m \in \mathbb{Z}, \exists k \in \mathbb{Z} : \forall n \in \mathbb{Z}, (m - n) | k$. This statement is true: given any $m \in \mathbb{Z}$, let $k = 0$. Then, for any $n \in \mathbb{Z}$, $(m - n) | k$. So the original statement – the negation of the statement just considered – is false.

5: Proofs. Complete the following exercises from S-POP: C(i)-1, C(i)-6, C(ii)-5, C(iii)-4.

C(i)-1. **Proposition.** (a) The sum of two odd numbers is even. (b) The product of two odd numbers is odd.

Proof. Let m and n be odd numbers. Then we can write $m = 2k - 1$ and $n = 2j - 1$, where $k, j \in \mathbb{Z}$.

(a) We have $m + n = (2k - 1) + (2j - 1) = 2k + 2j - 2 = 2c$, where $c = k + j - 1 \in \mathbb{Z}$. So $m + n \in 2\mathbb{Z}$; that is, $m + n$ is even.

(b) We have $m \cdot n = (2k-1) \cdot (2j-1) = 4kj - 2k - 2j + 1 = 2d + 1$, where $d = 2kj - k - j \in \mathbb{Z}$. So $m \cdot n \in 2\mathbb{Z} + 1$; that is, $m \cdot n$ is odd.

So the sum of two odd numbers is even, and the product of two odd numbers is odd.

ATWMR

C(i)-6. **Proposition.** If n is not divisible by 4, then n is not divisible by 12.

Proof (by contraposition). Assume that n is divisible by 12. Then $n = 12k$ for some $k \in \mathbb{Z}$. Since $12 = 4 \cdot 3$, we then have $n = 4 \cdot 3 \cdot k = 4c$, where $c = 3k \in \mathbb{Z}$. So n is divisible by 4.

So, by contraposition, if n is *not* divisible by 4, then n is *not* divisible by 12. **ATWMR**

C(ii)-5. **Proposition.**

$$\mathbb{Z} = 3\mathbb{Z} \cup 3\mathbb{Z} + 1 \cup 3\mathbb{Z} + 2.$$

Proof. We show that the set on each side of the above statement is contained in the set on the other side, as follows.

$\supseteq$) Let $x \in 3\mathbb{Z} \cup 3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$. Then, by definition of union, $x \in 3\mathbb{Z}$ or $x \in 3\mathbb{Z}+1$ or $x \in 3\mathbb{Z}+2$. In any of these cases, x is certainly an integer; that is, $x \in \mathbb{Z}$. So $3\mathbb{Z} \cup 3\mathbb{Z} + 1 \cup 3\mathbb{Z} + 2 \subseteq \mathbb{Z}$.

$\subseteq$) Let $x \in \mathbb{Z}$. Then, by the division algorithm, $\exists q, r \in \mathbb{Z}$ with $x = 3q + r$ and $0 \leq r < 2$. So $r = 0$, $r = 1$, or $r = 2$. In the first case we have $x = 3q \in 3\mathbb{Z}$; In the second case we have $x = 3q + 1 \in 3\mathbb{Z} + 1$; In the third case we have $x = 3q + 2 \in 3\mathbb{Z} + 2$. So in any case, we have $x \in 3\mathbb{Z} \cup 3\mathbb{Z} + 1 \cup 3\mathbb{Z} + 2$. So $\mathbb{Z} \subseteq 3\mathbb{Z} \cup 3\mathbb{Z} + 1 \cup 3\mathbb{Z} + 2$.

In sum, we've shown that

$$\mathbb{Z} = 3\mathbb{Z} \cup 3\mathbb{Z} + 1 \cup 3\mathbb{Z} + 2,$$

and we're done. **ATWMR**

C(iii)-4. **Proposition.** $\exists k \in \mathbb{Z}$ that can be expressed as a sum of two squares in two different ways.

Proof. Let $k = 50$. Then $k = 1^2 + 7^2 = 5^2 + 5^2$, so k can be expressed as a sum of two squares in two different ways. **ATWMR**

6: Symmetric difference. For this exercise you may use the definition, from pp. 6-7 of S-POP, of the symmetric difference $C \Delta D$ of two sets C and D. Fill in the blanks, to prove the following (when you hand in your exam, you can simply supply the words/phrases/symbols that complete the blanks, in order, separated by commas; you don't have to supply all of the surrounding text):

Proposition. Given any sets B, C, and D, we have

$$B \cap (C\Delta D) = (B \cap C)\Delta(B \cap D). \tag{0}$$

Proof. First of all note that, by S-POP, Proposition C(ii)-2_{E}, we have

$$(B \cap C)\Delta(B \cap D) = ((B \cap C) \cup (B \cap D)) - ((B \cap C) \cap \underline{\quad (B \cap D) \quad}). \tag{1}$$

But we proved in class that intersection distributes over union; that is, for sets B, C, D, we have

$$B \cap (\underline{\quad C \cup D \quad}) = (B \cap C) \cup (B \cap D). \tag{2}$$

Moreover, it's clear that the second B in $(B \cap C) \cap (B \cap D)$ is redundant, since the statement "$x \in B$ and $x \in C$, and $x \in B$ and $x \in D$" is clearly equivalent to "$x \in \underline{\quad B \quad}$ and $x \in C$ and $x \in D$." That is, we have

$$(B \cap C) \cap (B \cap D) = \underline{\quad B \quad} \cap C \cap D. \tag{3}$$

Plugging equations (2) and (3) into equation (1), and then plugging the result into equation (0), gives

$$B \cap (C\Delta D) = B \cap (C \cup D) - (B \cap C \cap D). \tag{4}$$

In other words: to prove (0), we need only prove (4). Let's do that.

$\subseteq$) Let $x \in B \cap (C\Delta D)$. Then by definition of ___intersection___, we have $x \in \underline{\quad B \quad}$ and $x \in C\Delta D$. From the latter statement and the fact that $C\Delta D = (C - D) \cup (D - C)$, we conclude that either $x \in C - D$ or $x \in \underline{\quad D - C \quad}$. We will consider these two cases separately; since one of these cases must hold, proving that the desired result holds in each of these two cases will prove that it holds in general.

Without loss of ___generality___, we may in fact assume that $x \in C - D$, because the case $x \in \underline{\quad D - C \quad}$ is the same, except that everywhere we see a C, we replace it with a $\underline{\quad D \quad}$, and vice versa.

So assume $x \in C - D$. This implies that $x \in C$ so, by definition of ___union___, certainly $x \in C \cup D$. Because we also know that $x \in B$, we therefore have $x \in \underline{\quad B \quad} \cap (C \cup D)$, by definition of ___intersection___. Moreover, since $x \notin \underline{\quad D \quad}$, we also have $x \notin B \cap C \cap D$, by definition of ___intersection___. So $x \in B \cap (C \cup D) - \underline{\quad B \cap C \cap D \quad}$.

Therefore, $\underline{\quad B \cap (C\Delta D) \quad} \subseteq B \cap (C \cup D) - B \cap C \cap D$.

$\supseteq$) Let $x \in B \cap (C \cup D) - B \cap C \cap D$. Then $x \in B$ and $x \in \underline{\quad C \cup D \quad}$ (by definition of ___intersection___), and $x \notin \underline{\quad B \cap C \cap D \quad}$. Now the fact that $x \in C \cup D$ implies, by definition of ____union____, that $x \in C$ ____or____ $x \in D$. We will consider these two cases separately. In fact, without loss of generality, we may assume that $x \in$ ____C____.

In this case we note that, since $x \in B$ as already noted, we have $x \in$ ____B____$\cap$____C____. But then since $x \notin B \cap C \cap D$, it must be that $x \notin$ ____D____. So $x \in C -$ ____D____. But then, by definition of ____union____, we have $x \in (C - D) \cup (D - C)$. That is, $x \in C\Delta D$, by definition of ____symmetric difference____. But again, we also have $x \in B$, so $x \in B \cap$ ____$(C\Delta D)$____, by definition of ____intersection____.

Therefore, $B \cap (C \cup D) - B \cap C \cap D \subseteq$ ____$B \cap (C\Delta D)$____.

We have thereby proved that equation (4) holds (by showing that the set on each side is a ____subset____ of the set on the other), and, as already noted, this proves our proposition.

____Your tag line____

(Supply your own tag line – that is, your own way of indicating the end of a proof – in the last blank.)