Take-Home Midterm Exam, due October 12

You are to complete this midterm on your own, without assistance from any human or other resources, except for: lectures notes from this class, T-BOP, and S-POP, yourself, and me (you are free to ask me questions, some of which I might not answer completely for you). You can also use paper and something to write with.

Complete this exam on your own paper. Be *neat*. **You must complete, and attach to your exam, a copy the cover sheet at the end of this exam.**

Your exam must be turned in at the *beginning* of class on Monday, October 12. Late exams will not be accepted.

Good luck!

1. Supply your own list of *at least five* shoulds and shouldnt's of mathematical communication. See S-POP, Part B, for more details. It's OK if one or more items on your list is something we have discussed in class. But be thoughtful. That is, your list should reflect things that are relevant to your own writing and thinking about mathematics.

2: Exercises from T-BOP: Complete the following exercises from the text.

(a) Section 1.7 (p. 23): 10, 12.

(b) Section 1.8 (p. 28): 13, 14.

3: Quantifiers. For this exercise, you might want to recall that the negation of a statement like "$\exists x \in X\colon Q(x)$" is the statement "$\forall x \in X,\ {\sim}Q(x)$." Here, ${\sim}Q(x)$ denotes the negation of $Q(x)$; that is, ${\sim}Q(x)$ means "not $Q(x)$." (In other words, ${\sim}Q(x)$ means "$Q(x)$ is false.") Similarly, the negation of a statement like "$\forall x \in X\colon Q(x)$" is the statement "$\exists x \in X, \sim Q(x)$."

(a) Let $Q(x, y)$ be a statement regarding objects x and y (in some universe U). How would you express the negation of the statement $\forall x \in X, \exists y \in Y\colon Q(x, y)$ in terms of ${\sim}Q(x, y)$? Hint: do this in stages, using the remarks made at the beginning of this exercise. That is: think of $\forall x \in X, \exists y \in Y\colon Q(x, y)$ as saying $\forall x \in X, P(x)$, where $P(x)$ is the statement $\exists y \in Y\colon Q(x, y)$. Now express ${\sim}(\forall x \in X, P(x))$ in terms of ${\sim}P(x)$, and then express ${\sim}P(x)$ in terms of ${\sim}Q(x, y)$.

(b) Express the negation of the statement $\exists x \in X\colon \forall y \in Y, Q(x, y)$ in terms of ${\sim}Q(x, y)$.

(c) Express the negation of the statement $\forall x \in X, \exists y \in Y\colon \forall z \in Z, Q(x, y, z)$ in terms of ${\sim}Q(x, y, z)$. (Here, $Q(x, y, z)$ is some statement involving objects x, y, z.)

4: More quantifiers. Identify each of the following statements as true or false (circle "**T**" or "**F**"). If a statement is true, explain why (you don't need to provide a complete proof; just a sentence or two will do). If a statement is false, provide a counterexample to the statement.

(a) $\forall m \in \mathbb{Z}$, $\forall n \in \mathbb{Z}$, $\exists k \in \mathbb{Z}$: $(m - n)|k$. **T** **F**

(b) $\exists k \in \mathbb{Z}$: $\forall m \in \mathbb{Z}$, $\forall n \in \mathbb{Z}$, $(m - n)|k$. **T** **F**

(c) $\sim\big(\forall m \in \mathbb{Z}$, $\exists k \in \mathbb{Z}$: $\forall n \in \mathbb{Z}$, $(m - n)|k\big)$. **T** **F**

5: Proofs. Complete the following exercises from S-POP: C(i)-1, C(i)-6, C(ii)-5, C(iii)-4.

6: Symmetric difference. For this exercise you may use the definition, from pp. 6-7 of S-POP, of the symmetric difference $C\Delta D$ of two sets C and D. Fill in the blanks, to prove the following (when you hand in your exam, you can simply supply the words/phrases/symbols that complete the blanks, in order, separated by commas; you don't have to supply all of the surrounding text):

Proposition. Given any sets B, C, and D, we have

$$B \cap (C\Delta D) = (B \cap C)\Delta(B \cap D). \tag{0}$$

Proof. First of all note that, by S-POP, Proposition C(ii)-2_{E}, we have

$$(B \cap C)\Delta(B \cap D) = ((B \cap C) \cup (B \cap D)) - ((B \cap C) \cap \underline{\hspace{3cm}}). \tag{1}$$

But we proved in class that intersection distributes over union; that is, for sets B, C, D, we have

$$B \cap (\underline{\hspace{3cm}}) = (B \cap C) \cup (B \cap D). \tag{2}$$

Moreover, it's clear that the second B in $(B \cap C) \cap (B \cap D)$ is redundant, since the statement "$x \in B$ and $x \in C$, and $x \in B$ and $x \in D$" is clearly equivalent to "$x \in$ __________ and $x \in C$ and $x \in D$." That is, we have

$$(B \cap C) \cap (B \cap D) = \underline{\hspace{2cm}} \cap C \cap D. \tag{3}$$

Plugging equations (2) and (3) into equation (1), and then plugging the result into equation (0), gives

$$B \cap (C\Delta D) = B \cap (C \cup D) - (B \cap C \cap D). \tag{4}$$

In other words: to prove (0), we need only prove (4). Let's do that.

$\subseteq$) Let $x \in B \cap (C\Delta D)$. Then by definition of __________________, we have $x \in$ __________ and $x \in C\Delta D$. From the latter statement and the fact that $C\Delta D = (C - D) \cup (D - C)$, we conclude that either $x \in C - D$ or $x \in$ ______________. We will consider these two cases separately; since one of these cases must hold, proving that the desired result holds in each of these two cases will prove that it holds in general.

Without loss of ________________, we may in fact assume that $x \in C - D$, because the case $x \in$ ______________ is the same, except that everywhere we see a C, we replace it with a __________, and vice versa.

So assume $x \in C - D$. This implies that $x \in C$ so, by definition of _____________, certainly $x \in C \cup D$. Because we also know that $x \in B$, we therefore have $x \in$ __________ $\cap (C \cup D)$, by definition of __________________. Moreover, since $x \notin$ __________, we also have $x \notin B \cap C \cap D$, by definition of __________________. So $x \in B \cap (C \cup D) -$ __________________.

Therefore, __________________ $\subseteq B \cap (C \cup D) - B \cap C \cap D$.

$\supseteq$) Let $x \in B \cap (C \cup D) - B \cap C \cap D$. Then $x \in B$ and $x \in$ ______________ (by definition of __________________), and $x \notin$ __________________. Now the fact that $x \in C \cup D$ implies, by definition of _____________, that $x \in C$ __________ $x \in D$. We will consider these two cases separately. In fact, without loss of generality, we may assume that $x \in$ __________.

In this case we note that, since $x \in B$ as already noted, we have $x \in$ __________ $\cap$ __________. But then since $x \notin B \cap C \cap D$, it must be that $x \notin$ __________. So $x \in C -$ __________. But then, by definition of _____________, we have $x \in (C - D) \cup (D - C)$. That is, $x \in C \Delta D$, by definition of __________________________. But again, we also have $x \in B$, so $x \in B \cap$ ______________, by definition of __________________.

Therefore, $B \cap (C \cup D) - B \cap C \cap D \subseteq$ __________________.

We have thereby proved that equation (4) holds (by showing that the set on each side is a _____________ of the set on the other), and, as already noted, this proves our proposition.

____

(Supply your own tag line – that is, your own way of indicating the end of a proof – in the last blank.)

(Don't forget the cover page below.)

MATH 2001-004: Intro to Discrete Math

September 28, 2015

TAKE-HOME MIDTERM EXAM

I have neither given nor received unauthorized assistance on this exam.

Name:____

Signature:____

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	15 pts	
2	18 pts	
3	15 pts	
4	15 pts	
5	20 pts	
6	17 pts	
TOTAL	100 pts	