1. **Cool Fact about exponential functions.** Suppose some quantity Q is subject to exponential growth or decay. Then Q behaves in an interesting way: if Q changes by a factor F over some time period of length T, then it changes by this *same* factor F over the *next* period of length T, and again by the *same* factor over the *next* such period, and so on.

   Fill in all of the blanks on this page (there are fourteen of them).

   For example: if Q doubles after 6 years, then after *another* 6 years, it will double again (so that, after 12 years, there is _____ times as much as there was at very beginning); after another 6 years, it will double again (so that, after 18 years, there is _____ times as much as there was at very beginning); etc. As another example: if Q decays by a factor of 5 after 7 hours, then after *another* 7 hours, it will decay by a factor of 5 again (so that, after 14 hours, there is ________ times as much as there was at very begining); after another 7 hours, it will decay by a factor of _____ again (so that, after 21 hours, there is _________ times as much as there was at very beginning); etc.

   We've discussed this in class; now we demonstrate exactly why this is so. We consider exponential growth only, but the arguments are similar in the case of exponential decay.

   Suppose $Q(t)$ satisfies the exponential growth initial value problem

   $$\frac{dQ}{dt} = kQ, \quad Q(0) = Q_0.$$

   Then the formula for $Q(t)$ at time t is given by

   $$Q(t) = Q_0 \underline{\hspace{2cm}}. \tag{1}$$

   So, from time $t = 0$ to time $t = T$ (where T is any real number), we see that the value of Q changes from $Q(0) =$ ______ to $Q(T) = Q_0 e^{kT}$. So $Q(T)/Q(0) = e^{kT}$, meaning that, over this interval, Q grows by a factor of _______.

   But now, we note the following: over the next interval of the same duration, from $t = T$ to time $t = 2T$, the value of Q changes from $Q(T) = Q_0 e^{kT}$ to $Q(2T) =$ __________. So $Q(2T)/Q(T) = (Q_0 e^{2kT})/(Q_0 e^{kT}) =$ _______, meaning that, over this new interval of length T, Q again grows by a factor of e^{kT}.

   Similarly (one more time): over the next interval of the same duration, from $t = 2T$ to time $t = 3T$, the value of Q changes from $Q(2T) =$ __________ to $Q(3T) = Q_0 e^{3kT}$. So $Q(3T)/Q(2T) = (Q_0 e^{3kT})/(Q_0 e^{2kT}) =$ _______, meaning that, over this new interval of length T, Q again grows by a factor of _______.

   And so on. To summarize: given an an exponentially growing quantity Q, the value of $Q(t)$ will increase by the same factor over each successive interval of a given length. (again, a similiar thing happens for ______________ decay.)

2. A certain culture of bacteria grows in such a way that its rate of growth is proportional to the size $B(t)$ of the culture. Suppose, after 5 hours, there is 3 times as much as there was initially.

   (a) After how long will there be 27 times as much as there was initially? Answer without writing down any formulas; just use the reasoning of exercise 1 above.

   (b) Without writing down any formulas, estimate how long it will take before there is 100 times as much as there was initially. If possible, state your answer to the nearest five hours.

   (c) Write down a formula for $B(t)$. Your formula may involve unknown constants k and B_0.

(d) In your formula from part (c) above, solve for k, and put this value of k back into your formula. (Do not simplify yet.)

(e) Now use your formula from part (d) above (and a calculator) to answer parts (a) and (b) above. (For part (b), this time obtain an exact answer, not just an estimate.)

(f) Simplify your answer in part (d) above, to an obtain an expression for $B(t)$ that involves an exponential function with base 3.

(g) Use your formula from part (f) above to again answer part (a) above. Hint: set $B(t)$, as expressed in part (f) above, equal to $27B_0$. Divide by B_0 to get an equation with a power of 3 on each side. Set the exponents equal and solve for t.