MATH 1310–001: CLS
Second Take-Home Exam
("All About Arctan")
Due Sunday, October 25 at 10:00 PM

Though I may have made use of outside resources for this exam, everything I have written here reflects my own understanding of the material, and is written in my own words.

Name: Dr. Slam

Signature: Eric Steele

Please read the instructions on the next page **CAREFULLY.**

GOOD LUCK!!

Question:	1	2	3	4	5	6	7	8	Total
Points:	4	11	17	16	15	16	6	15	100
Score:									

DIRECTIONS

This exam is OPEN EVERYTHING! You can use any notes, electronic/online sources, or human resources you would like. But you must UNDERSTAND what you write in the end, and state it in your own words (and math symbols).

You must **show your work** on every problem of this exam. You will be graded on the quality of your exposition and reasoning! Please write everything out using complete sentences, careful arguments, proper mathematical notation, and so on. Please provide the specified number of decimal places in your answers, where requested.

Please complete all work in the space provided. It is fine to print this exam out and complete it by hand. I recommend using scratch paper until you have a solution you're happy with, and then writing that solution carefully on these pages.

If you have access to a word processing program that is good with math symbols, and you have the technology and expertise to complete this exam that way, that's fine too.

Either way, please turn the exam in through Canvas. (Go to "Assignments" on our Canvas page, Click on "Exam 1, October 25" under "Upcoming assignments," and follow the directions there.) If you have written out your exam by hand, then you can submit either a scan of your exam, or photos of the pages, taken with your phone/tablet/computer. Just MAKE SURE IT'S LEGIBLE or you will lose points.

If you get stuck on a problem, please skip it and then come back. You have plenty of time!!

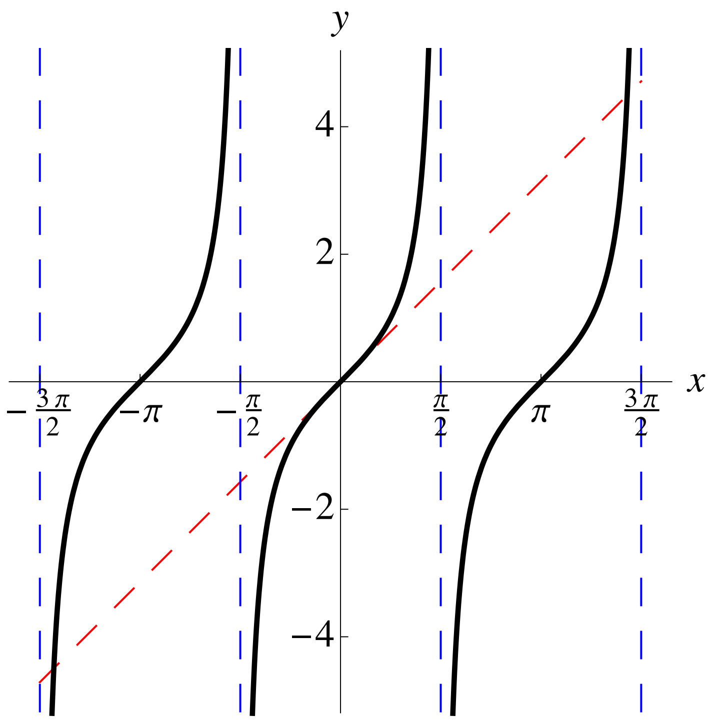
Please sign below:

I have read and I understand these directions. ______________________________

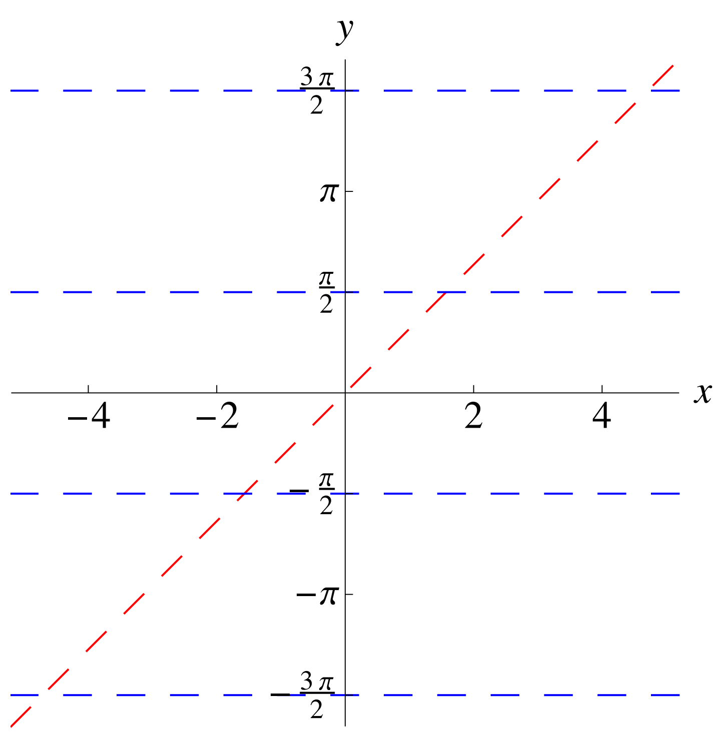
1. (4 points) Make sure you turn in BOTH signature pages with your exam, or you will lose points!!!

!!

2. On the axes directly below is the graph of $f(x) = \tan(x)$, on the domain $(-3\pi/2, 3\pi/2)$. The line $y = x$ is dashed in, in red, for reference. The lines $x = \pm\pi/2$ and $x = \pm 3\pi/2$ are also dashed in, in blue, for reference. (None of the dashed lines are part of the function $f(x) = \tan(x)$ itself; they are just "guidelines" for helping visualize how the function behaves.)



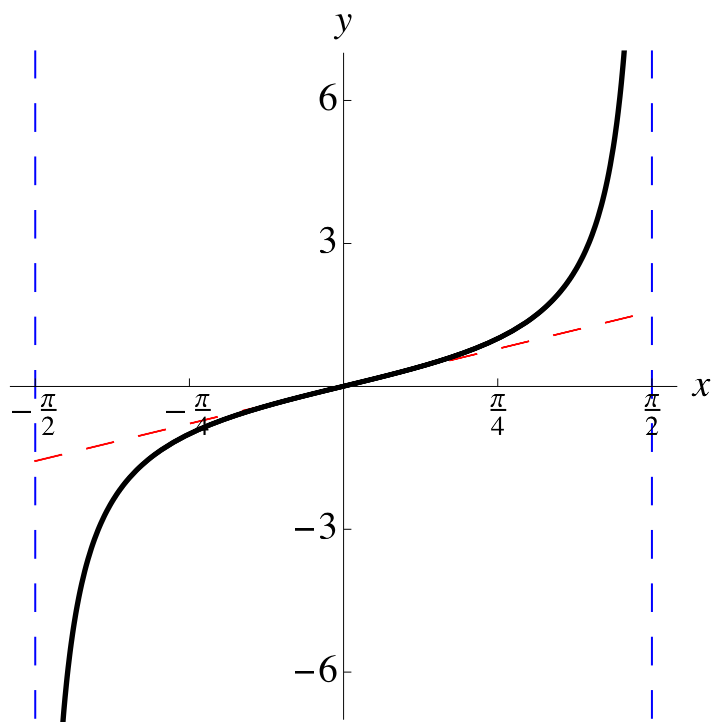
(a) (6 points) Carefully draw, on the axes below, what you get by reflecting (that is, taking the mirror image of) the above function, on the given domain, about the line $y = x$.



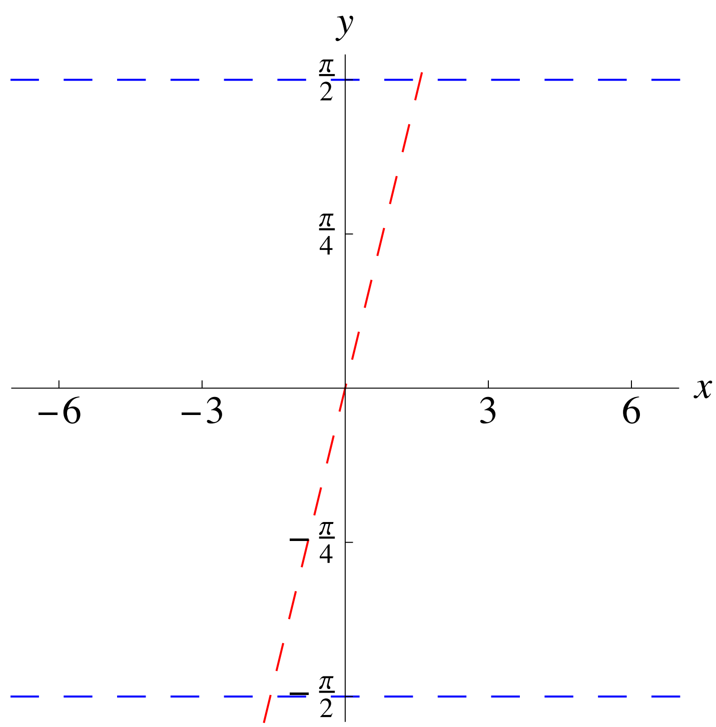
(b) (5 points) Does the graph that you supplied on the previous page give y as a function of x? Please explain completely. You can refer to certain rules that you may have learned in previous math classes (or that you may have found on the web).

But an answer like "This is true by such and such a rule" is insufficient. You must also state the rule carefully and completely, and describe how it applies to the current situation.

3. On the axes directly below is the graph of $f(x) = \tan(x)$, on the domain $(-\pi/2, \pi/2)$. The line $y = x$ is dashed in, in red, for reference. The lines $x = \pm\pi/2$ are also dashed in, in blue, for reference. (None of the dashed lines are part of the function $f(x) = \tan(x)$ itself; they are just "guidelines" for helping visualize how the function behaves.)



(a) (6 points) Carefully draw, on the axes below, what you get by reflecting (that is, taking the mirror image of) the above function, on the given domain, about the line $y = x$.



(b) (5 points) Explain completely why the graph that you supplied on the previous page **does** give y as a function of x. You can refer to certain rules that you may have learned in previous math classes (or that you may have found on the web).

But an answer like "This is true by such and such a rule" is insufficient. You must also state the rule carefully and completely, and describe how it applies to the current situation.

Use the Vertical Line Rule
(or Test)

But make sure you answer completely!!

(c) (6 points) The graph that you supplied at the bottom of Page 5 is called the *arctangent* function. That is, you have just graphed the function

$$g(x) = \arctan(x).$$

Fill in the blanks: The function $g(x) = \arctan(x)$ is what you get by reflecting the graph of the function $f(x) =$ __________________, on the domain _______________________, about the line __________________.

4. Whenever we find a new function, like $\arctan(x)$, we would like to know its derivative. (This IS calculus, after all.) To this end we will show, in this problem, that

$$\boxed{\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}.}$$

Here's how: let's consider a point (a,b) on the graph of $f(x) = \tan(x)$. Let's look at what happens to that point, and to the tangent line to the graph at that point, when we reflect everything about the line $y = x$.

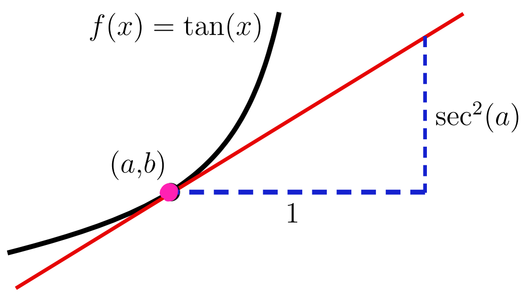
(a) (2 points) Fill in the blank: first of all, since the point (a,b) is on the graph of $f(x) = \tan(x)$, we know that

$$b = \tan(\underline{\quad a \quad}). \tag{1}$$

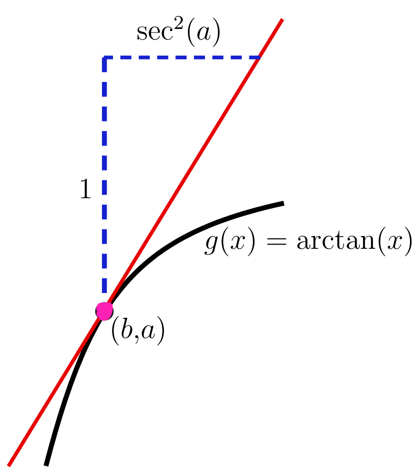
(b) (2 points) Fill in the blank:
since

$$\frac{d}{dx}[\tan(x)] = \sec^2(x),$$

we know that the slope of the <u>tangent</u> line to the graph of $f(x) = \tan(x)$, at the point (a,b) on that graph, is equal to $\sec^2(a)$.
Here's a picture:



(c) (4 points) Recall that reflecting the graph of $f(x) = \tan(x)$ (for $-\pi/2 < x < \pi/2$) about the line $y = x$ gives us the graph of $g(x) = \arctan(x)$. Also, reflecting the point (a,b) about the line $y = x$ gives us the point (b,a). So here's what we get if we reflect the above picture about the line $y = x$:



By looking at the above picture and recalling that slope=rise/run, fill in two two blanks: The slope of the tangent line to the graph of $g(x) =$ arctan(x), at the point (b,a) on that graph, is equal to 1 over $\sec^2(a)$. Or, in short, since the slope of the tangent line equals the derivative:

$$g'(b) = \frac{1}{\sec^2(a)}. \tag{2}$$

(d) (2 points) On the other hand, we have a trig identity that tells us

$$\sec^2(\theta) = 1 + (\tan(\theta))^2$$

for any real number θ (provided both sides exist). So we can conclude from equation (2) above that (fill in the blank)

$$g'(b) = \frac{1}{1 + (\tan(a))^2}. \tag{3}$$

(e) (6 points) Recall from equation (1) above that $\tan(a) =$ b. Plugging this in to the right hand side of (3) gives (fill in the blank)

$$g'(b) = \frac{1}{1 + (b)^2}, \tag{4}$$

or, if we simply replace every b in (4) with an x,

$$g'(x) = \frac{1}{1 + x^2}.$$

So we're done: we've shown that the derivative of $g(x) = \arctan(x)$ is given by

$$g'(x) = \frac{1}{1+x^2} \quad \text{(fill in the blank).}$$

5. (a) (5 points) Graph the function $f(x) = \dfrac{1}{1+x^2}$ using Sage. Please supply a copy of the graph with this exam.

(b) (10 points) Fill in the blanks – in each blank, the correct answer is the word "sigmoid" (which means "roughly S-shaped," or actually more like "elongated S-shaped"), or the word "bell," or the mathematical expression "bI."

The arctangent function has a ____________________ shape, and we've just seen that its derivative has a ________________ shape.

OK, so, the derivative of a ____________________ curve is a ________________ curve. Where have we seen this before? We've seen it in SIR!! Remember that, there, the variable R (recovered) followed a ____________________ curve, while the variable I (infected) followed a ________________ curve. But I *is* the derivative of R, or more precisely I is *proportional to* the derivative of R, by the third of the SIR equations, which says $R' =$_______________. That is: the derivative of the ____________________ curve R *is* the ________________ curve _______________.

6. Now that you know how to differentiate $\arctan(x)$ (see the boxed equation at the top of Page 8), compute the following derivatives. You do not need to simplify your answers, but do show your work.

(a) (4 points) $f'(x)$ if $f(x) = \arctan(e^{3x})$

$$f'(x) = \frac{1}{1+(e^{3x})^2} \cdot \frac{d}{dx}\left[e^{3x}\right]$$

$$= \frac{3e^{3x}}{1+(e^{3x})^2} = \frac{3e^{3x}}{1+e^{6x}}$$

(b) (4 points) $\frac{d}{dq}\left[\cos(q)\arctan(q)\right]$

(c) (4 points) $\frac{dy}{dx}$ if $y = \frac{\arctan(x)}{x}$

$$\frac{dy}{dx} = \frac{x \cdot \frac{1}{1+x^2} - \arctan(x)}{x^2}$$

(d) (4 points) $\frac{d}{dx}\left[e^{3\arctan(x)}\right]$.

7. (6 points) (A function whose derivative is $\arctan(x)$.) Let

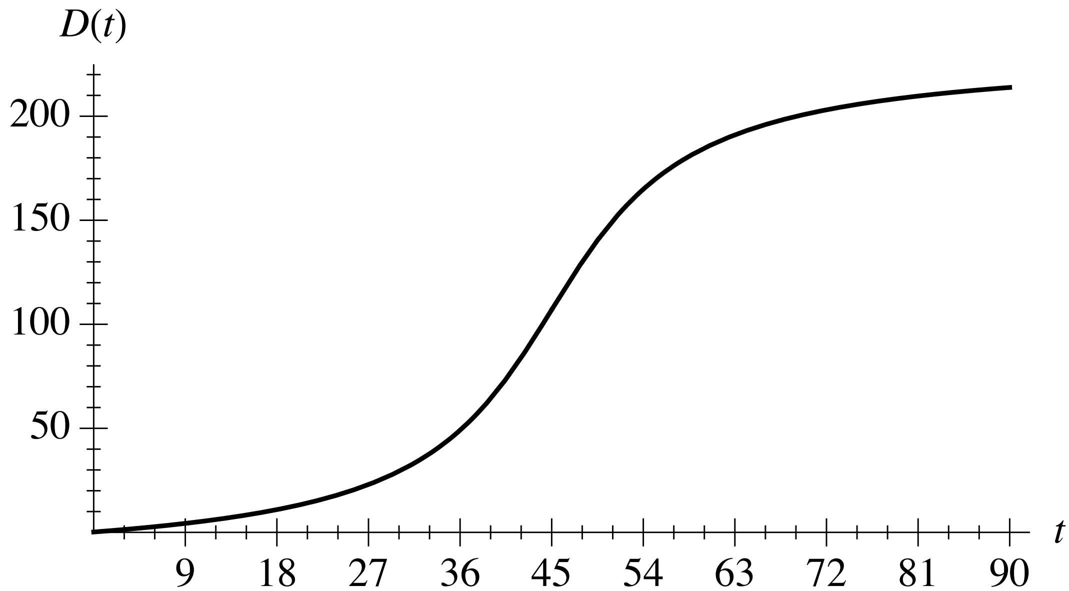
$$h(x) = x \arctan(x) - \frac{1}{2} \ln(1 + x^2).$$

Show that

$$h'(x) = \arctan(x).$$

8. The graph below shows a function $D(t)$ given by the formula

$$D(t) = 79\arctan(0.1(t-45)) + 107.$$



We are now going to consider the relevance of this graph to an actual 90-day-long outbreak of Ebola in the Democratic Republic of Congo (DRC), in 1995.

Here is a table of deaths due to this outbreak. The top row of the table breaks up the ninety-day period into three-day increments. The bottom row denotes the total, or *cumulative*, number of deaths $C(t)$, due to this Ebola outbreak, from day 0 to the end of day t.

Time t (days)	3	6	9	12	15	18	21	24	27	30
Total deaths $C(t)$	1	1	1	4	11	13	13	17	24	33
Time t (days)	33	36	39	42	45	48	51	54	57	60
Total deaths $C(t)$	33	38	52	88	112	121	135	153	170	182
Time t (days)	63	66	69	72	75	78	81	84	87	90
Total deaths $C(t)$	187	193	195	204	208	210	211	214	214	214

(a) (5 points) Directly on top of the graph of $D(t)$ on this page, plot the points given in the above table. That is: plot (with a small circle or dot) each of the points $(t,C(t))$, where $t = 3,6,9,\ldots,90$. (You don't need to connect the dots.)

(b) (2 points) Do you believe that the arctangent function has relevance to the study of disease? Justify your answer.

(c) (5 points) Let $D(t)$ be the function defined on the previous page. Show carefully that

$$D'(t) = \frac{7.9}{1 + 0.01(t-45)^2}.$$

$$D'(t) = \frac{d}{dt}\left[79\arctan(0.1(t-45)) + 107\right]$$

$$= \frac{d}{dt}\left[79\arctan(0.1(t-45))\right] + \frac{d}{dt}[107]$$

0

$$= 79\,\frac{d}{dt}\left[\arctan(0.1(t-45))\right]$$

$$= 79 \cdot \frac{1}{1+(0.1(t-45))^2} \cdot \frac{d}{dt}\left[0.1(t-45)\right]$$

$$= \frac{79}{1+0.01(t-45)^2} \cdot 0.1$$

= (go ahead and finish).

(d) (3 points) Consider the Ebola outbreak studied in the previous parts of this problem. Instead of looking at *cumulative* deaths $C(t)$, let's now consider the *death rate* $R(t)$, measured in deaths per day. Explain why it might be reasonable to model $R(t)$ by the formula

$$R(t) = \frac{7.9}{1 + 0.01(t-45)^2}.$$

Hint: look at part (c) of this problem, on Page 14.