Modeling with IVP's: examples.
Today: population growth.

(A) Exponential (unlimited) growth (recap).
Here, the rate of growth of the population is proportional to the population size:

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0. \qquad (*)$$

The parameter $k > 0$ is the per capita growth rate.
Interpretation:
write the DE in $(*)$ as

$$k = \frac{dP/dt}{P}.$$

So k is rate of growth of population per unit of population.
E.g. if P is in rabbits and t in months, then k has units

$$\frac{\text{rabbits/month}}{\text{rabbit}} \quad \text{or} \quad \text{month}^{-1}:$$

k tells us how many rabbits per month result from each rabbit.
We've seen that $(*)$ has closed-form solution

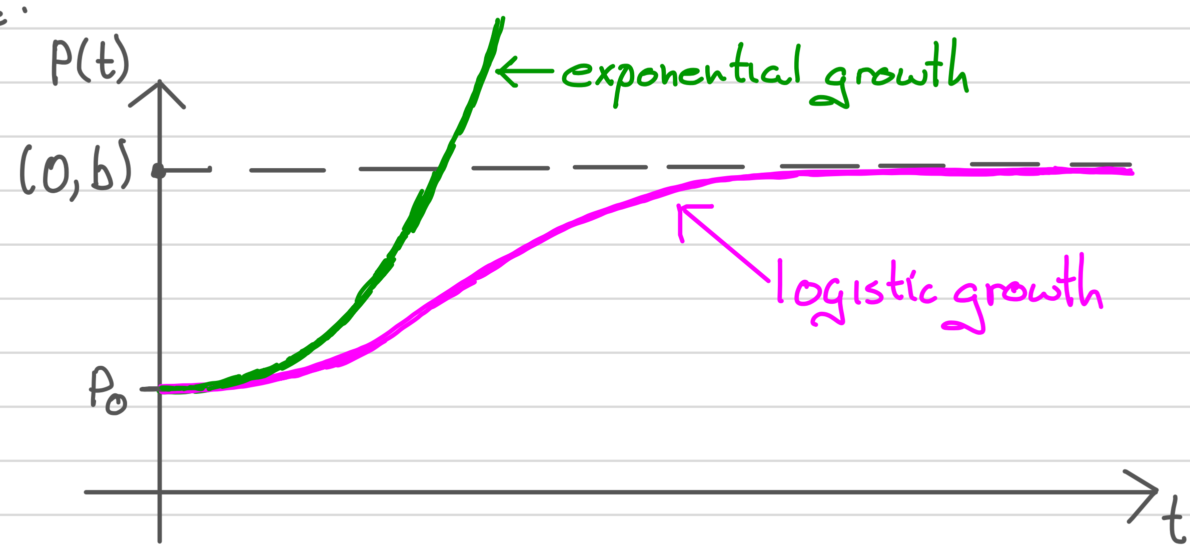
$$P(t) = P_0 e^{kt}.$$

(B) Logistic (bounded) growth.

If an environment can support only a finite number - say, b - of rabbits, then instead of $(*)$, we might have

$$\frac{dP}{dt} = \underbrace{k}_{\text{"per capita growth rate"}} P\left(1 - \frac{P}{\underbrace{b}_{\text{"carrying capacity"}}}\right), \qquad P(0) = P_0. \qquad (*')$$

Picture:



One can show that $(*')$ has closed-form solution

$$P(t) = \frac{P_0 b e^{kt}}{b + P_0(e^{kt} - 1)}.$$

We've also studied $(*')$ using Euler's method.

(c) Dual populations, e.g. predator/prey.

Say we have rabbits R and foxes F. We assume:

(i) logistic growth of R, in the absence of F;

(ii) rabbits die (through predation by F) at a rate proportional to $R \cdot F$;

(iii) Foxes are born at a rate proportional to $R \cdot F$;

(iv) Foxes die at a rate proportional to F.

These assumptions yield the "Lotka-Volterra equations:"

$$(LV) \begin{cases} \dfrac{dR}{dt} = \overbrace{aR\left(1-\dfrac{R}{b}\right)}^{(i)} - \overbrace{cRF}^{(ii)} \\ \dfrac{dF}{dt} = \underbrace{dRF}_{(iii)} - \underbrace{eF}_{(iv)} \end{cases}$$

(a, b, c, d, e are positive parameters).

There are no closed-form solutions to (LV); we need to use numerical methods to solve them.

Some possible tweaks to (LV):

(a) Replace term (i) with kR for some $k>0$ (that is, assume unlimited growth of rabbits);

(b) The "May Model" entails the following changes to (LV):

- replace cRF (term (ii)) with $\dfrac{cRF}{R+m}$

for some parameter $m>0$. This reflects the fact that no fox can eat an unlimited number of rabbits. (Why? Well, cRF just keeps growing as R does; the denominator $R+m$ slows this growth.)

- replace dRF (term (iii)) with

$$dF\left(1-\frac{F}{pR}\right).$$

That is: foxes grow logistically, with carrying capacity proportional to the number of rabbits present.

Next time: more IVP's.