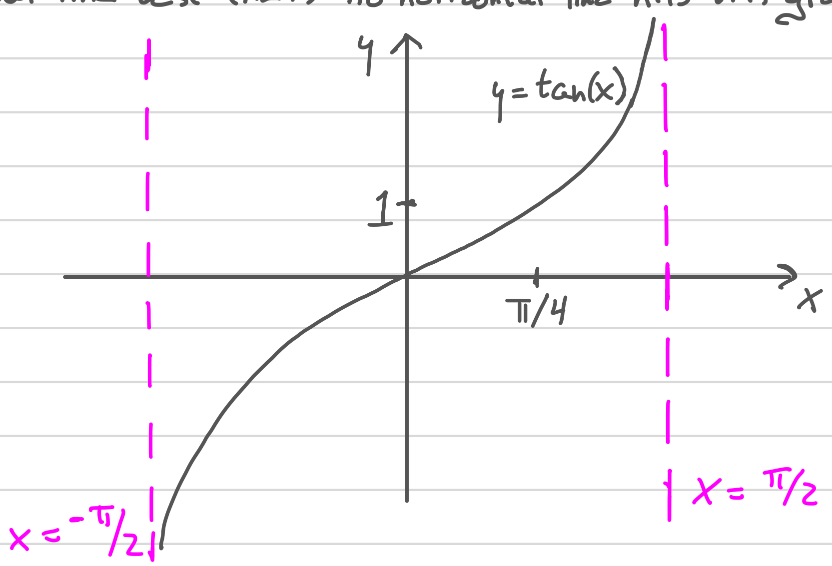
The arctangent function.

GOALS: by flipping (reflecting) $y = \tan(x)$ about $y = x$, we'll get a new function $y = \arctan(x)$ that:

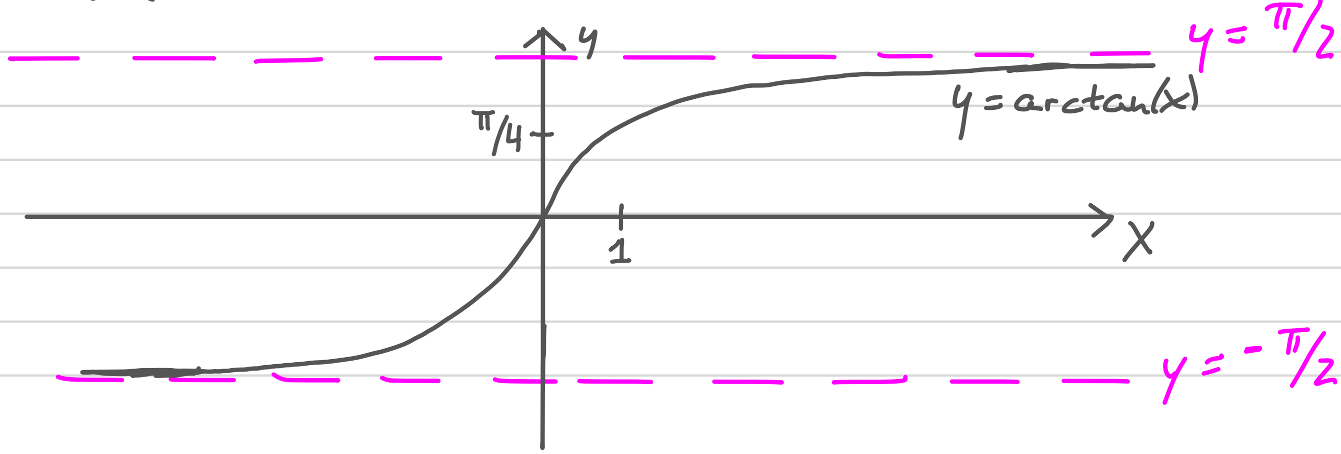
(1) has a sigmoid ("S") shape;
(2) has a bell-shaped derivative;
(3) models certain diseases.

(I) "Inverting" $y = \tan(x)$.

Note that the portion of $y = \tan(x)$ with domain $(-\pi/2, \pi/2)$ satisfies the horizontal line test (HLT): no horizontal line hits this graph more than once.



So flipping the above graph gives a graph that satisfies the vertical line test, and is therefore the graph of a function. Let's call this function $y = \arctan(x)$:



Goal (1): achieved. ✓

II) The derivative.

Note that $y = \arctan(x)$ takes x to $\arctan(x)$. Its "flip" $y = \tan(x)$ reverses the roles of input and output, and therefore takes $\arctan(x)$ to x. That is,

$$\boxed{\tan(\arctan(x)) = x.} \qquad \text{(FLAP)}$$

Differentiate both sides of (FLAP); by the chain rule on the left, we get

$$\sec^2(\arctan(x)) \frac{d}{dx}[\arctan(x)] = 1.$$

Divide by $\sec^2(\arctan(x))$:

$$\frac{d}{dx}[\arctan(x)] = \frac{1}{\sec^2(\arctan(x))}. \qquad (*)$$

Now simplify as follows: the trig identity $\sec^2(\theta) = 1 + \tan^2(\theta)$ tells us that

$$\sec^2(\arctan(x)) = 1 + \tan^2(\arctan(x)) = 1 + (\tan(\arctan(x)))^2 \underset{\text{by (FLAP)}}{=} 1 + x^2.$$

So $(*)$ gives

$$\boxed{\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}.} \qquad \text{Differentiation formula (H)}$$

FACT: Goals (2) + (3) are now achieved. See tomorrow's tutorial.

Examples.

Find:

(1) $\frac{d}{dx}[\arctan(5x^3)]$, (2) $\frac{d}{dy}[5\arctan^3(y)]$, (c) $\frac{d}{dz}[(1+z^2)\arctan(z) - z]$.

Solution.

(1) $\frac{d}{dx}[\arctan(5x^3)] = \frac{1}{1+(5x^3)^2} \cdot \frac{d}{dx}[5x^3] = \frac{15x^2}{1+25x^6}.$

(2) $\frac{d}{dy}[5\arctan^3(y)] = 5\frac{d}{dy}[\arctan^3(y)] = 5 \cdot 3\arctan^2(y) \cdot \frac{d}{dy}[\arctan(y)]$

$= \frac{15\arctan^2(y)}{1+y^2}.$

(3) $\frac{d}{dz}[(1+z^2)\arctan(z) - z]$

$= (1+z^2) \cdot \frac{d}{dz}[\arctan(z)] + \arctan(z) \cdot \frac{d}{dz}[1+z^2] - 1$

$= \frac{1+z^2}{1+z^2} + \arctan(z) \cdot 2z - 1 = 1 + 2z\arctan(z) - 1$

$= 2z\arctan(z).$