Exponential growth and decay: modeling.

Last week, we saw that the exponential growth/decay initial value problem (IVP) given by

(growth) $\frac{dP}{dt} = kP, \quad P(0) = P_0$ | $\frac{dR}{dt} = -kR, \quad R(0) = R_0$ (decay)

has solution

$P(t) = P_0 e^{kt}$ | $R(t) = R_0 e^{-kt}$.

Here, P_0/R_0 is the initial value of P/R, and k is the per capita growth rate/per unit decay rate.

Example 1. A certain exponentially growing population doubles every three years.

(a) What's the per capita growth rate k?

(b) If $P_0 = 10$ (in millions), find:

(i) An explicit formula for $P(t)$.

(ii) When $P(t) = 100$ (million).

Solution. We have

$$P(t) = P_0 e^{kt} \qquad (*)$$

for certain constants k and P_0.
We're given that

$$P(3) = 2P_0,$$

so by $(*)$,

$$P_0 e^{k \cdot 3} = 2P_0.$$

Solve for k:

$$e^{3k} = 2$$
$$\ln(e^{3k}) = \ln(2)$$
$$3k = \ln(2)$$
$$k = \ln(2)/3 = 0.231049 \quad \text{year}^{-1}$$

(b)(i) $P(t) = P_0 e^{kt} = 10e^{0.231049t}$.

(ii) We solve

$$P(t) = 100$$

for t:

$$10e^{0.231049t} = 100$$
$$e^{0.231049t} = 10$$
$$\ln(e^{0.231049t}) = \ln(10)$$
$$0.231049t = \ln(10)$$
$$t = \ln(10)/0.231049 = 9.96579 \text{ years.}$$

Remarks.

(A) We saw, in part (a) of the example, that $k = \ln(2)/3$. So the answer to part (b)(i) can be rewritten:

$$P(t) = 10e^{(\ln(2)/3)t} = 10\left(e^{\ln(2)}\right)^{t/3} = 10 \cdot 2^{t/3}.$$

This clearly shows that $P(t)$ doubles every 3 years (since adding 3 to t causes $2^{t/3}$ to double).

(B) "Gut check:" since $P(t)$ doubles every three years, we can "chart" its growth like this:

P(t) = 100 in here somewhere

$$P_0 = 10 \xrightarrow{\text{3 years}} 20 \xrightarrow{\text{3 years}} 40 \xrightarrow{\text{3 years}} 80 \xrightarrow{\text{3 years}} 160$$

which confirms that $P(t) = 100$ after between 9 and 12 years. This does not prove that (b)(ii) is correct, but it does tell us we're in the right ballpark.

<u>Example 2.</u> Salt dissolves in water at a rate proportional to the amount $S(t)$ remaining.

If 6 lb. of salt reduces to 5 lb. after 1 hr, how much remains after 4 hrs?

<u>Solution.</u> We have

$$\frac{dS}{dt} = -kS, \qquad S(0)=6,$$

so by our exponential decay IVP above,

$$S(t) = 6e^{-kt}. \qquad (**)$$

To find k note that $S(1) = 5$, so by $(**)$,

$$6e^{-k\cdot 1} = 5$$
$$e^{-k} = 5/6$$
$$-k = \ln(5/6)$$
$$k = -\ln(5/6) = 0.182321t$$

So again by $(**)$,

$$S(t) = 6e^{-0.182321t}.$$

So $$S(4) = 6e^{-0.182321\cdot 4} = 2.893543 \text{ lb.}$$

<u>Note.</u>

We could also write our formula for $S(t)$ this way:

$$S(t) = 6e^{-kt} = 6e^{-(-\ln(5/6))t} = 6e^{\ln(5/6)t} = 6\left(e^{\ln(5/6)}\right)^t = 6\cdot(5/6)^t.$$

This reflects how the salt reduces by a factor of $5/6$ each hour:

$$S_0 = 6 \xrightarrow{1\text{ hr}} 6\cdot\tfrac{5}{6} \xrightarrow{1\text{ hr}} 6\cdot\tfrac{5}{6}\cdot\tfrac{5}{6} \xrightarrow{1\text{ hr}} 6\cdot(5/6)^3 \xrightarrow{1\text{ hr}} 6\cdot(5/6)^4 \ (= 2.893519)$$