For this Activity, we will need the Sage program Population_growth.sws.

1. Constant per capita (also known as unlimited or exponential) growth

This question considers the unlimited growth initial value problem:

$$\frac{dR}{dt} = 0.1R; \qquad R(0) = 2{,}000$$

(R is in rabbits; dR/dt is in rabbits per month).

Run the program Population_growth.sws, which will produce, on the same set of axes:

(i) A graph (in blue circles) of an approximate solution (obtained using Euler's method) to the above initial value problem; and

(ii) A graph (in red) of the exact, closed form solution (that is, the solution given by a formula) to the above initial value problem.

Answer these questions:

(a) Write down the formula for the exact solution to the above IVP. (Use what you know about the unlimited growth IVP.)

$$R = 2{,}000e^{0.1t}.$$

(b) What *single quantity* in your program Population_growth.sws would you change, to make the numerical and closed-form solutions agree more closely? Go ahead and make that change to your program, and run it again to make sure things worked. Continue to adjust as necessary until the two solutions appear to fit each other as closely as possible.

Decrease the stepsize. Putting stepsize=0.01 seems to do the trick.

(c) What happens to a population of 2,000 rabbits after 6 months, and after 2 years? Read the answers as well as you can from the graph, and then check your work using your exact solution.

$$R(6) = 2{,}000e^{0.1\cdot 6} = 3{,}644.24 \text{ rabbits}; \quad R(24) = 2{,}000e^{0.1\cdot 24} = 22{,}046.35 \text{ rabbits.}$$

(d) How long does it take the rabbit population to reach 25,000? Read the answer as well as you can from the graph, and then check your work using your exact solution.

From the graph, it looks like it takes about 25 months.

We check this by solving $R(t) = 25{,}000$ for t:

$$\begin{aligned} 2{,}000e^{0.1t} &= 25{,}000 \\ e^{0.1t} &= 12.5 \\ 0.1t &= \ln(12.5) \\ t &= \ln(12.5)/0.1 = 25.2573 \end{aligned}$$

months.

2. Logistic growth

The following questions concern a rabbit population described by the *logistic* differential equation

$$\frac{dR}{dt} = 0.1R\left(1 - \frac{R}{25{,}000}\right)$$

rabbits per month. We'll study this model in class very soon, but for now, we'll do the following.

(a) For this exercise, we will need to *modify* the program Population_growth.sws, so that it solves this logistic growth problem, instead of the unlimited growth problem on the previous pages. Here are some hints for doing the modification:

- ☐ We'll need to change the quantity `tfin`, so that your graph goes out far enough in time to see what happens after 5 years.
- ☐ We'll need to change the formula for `Rprime` to model logistic growth, as given by the above differential equation.
- ☐ We'll need to delete the line

  ```
  Tplot=plot(2000*e^(0.1*x),0,30,color='red',thickness=3)
  ```

  from your program.
- ☐ We'll need to delete the part that says `+Tplot` from the last line of your program.

(Your program should retain the same stepsize that you ended up with in exercise 1(b) above.) Once you have made the above modifications, run the program. You should get a graph that starts out looking something like your curve in Exercise 1 of this activity, but that "levels off" as you go out in time.

Now answer these questions:

i. Under this logistic growth model, what happens to a population of 2,000 rabbits after 6 months, after 2 years, and after 5 years? Read these values off the graph as well as you can.
From the graph we find that, very roughly, $R(6) = 3{,}500$ rabbits; $R(24) = 12{,}000$ rabbits; $R(60) = 24{,}000$ rabbits.

ii. Edit your program so that your starting number of rabbits is now 40,000 instead of 2,000. Compare your new graph to the previous logistic graph. How do the graphs differ? In what ways are they similar?
The new graph is quite different from the previous one. In particular, the new R is decreasing, whereas the previous R was increasing. The two graphs are similar in that, in both cases, R levels off at 25,000.

iii. Fill in the blanks – each blank should be filled in with one of these words:

less greater increasing decreasing positive negative

In the above logistic growth situation, the population will increase if the initial population is ___less___ than 25,000, and will decrease if the initial population is ___greater___ than 25,000. To see this mathematically, look at the logistic differential equation above. Note that, if the initial population is less than 25,000, then the quantity $R/25{,}000$ in the logistic equation will initially be ___less___ than 1, so the derivative

$$\frac{dR}{dt} = R\left(1 - \frac{R}{25{,}000}\right)$$

will initially be positive, meaning R will initially be ___increasing___. On the other hand, if the initial population is greater than 25,000, then the quantity $R/25{,}000$ will initially be ___greater___ than 1, so $\frac{dR}{dt}$ will initially be ___negative___, so R will initially be ___decreasing___.