Week 6 - Wednesday, 2/20

Logarithms: an introduction.

A) Motivating example.

It's known that radium 226 decays according to

$$\frac{dR}{dt} = -kR, \qquad R(0) = R_0,$$

where $k = 1/2337$ year^{-1} is the decay rate and R_0 the initial amount.

Question: what's the "half-life" τ of radium 226, meaning: how long does it take for a sample to decay by half?

Solution. We know, by (Ed) of 2/18, that

$$(*) \qquad R(t) = R_0 e^{-kt}.$$

We want to find τ so that $R(\tau) = \frac{1}{2}R_0$, or, by $(*)$,

$$R_0 e^{-k\tau} = \frac{1}{2}R_0.$$

To solve for τ, we divide by R_0:

$$e^{-k\tau} = 1/2.$$

But now, how do we "get τ out of the exponent" to solve for τ?

COOL FACT: The answer lies in a function we've seen already!!

Specifically, we have:

B) The natural logarithm function.

Recall that, if $b>0$, then $\ln(b)$ is the number such that

$$\frac{d}{dx}[b^x] = \ln(b)\,b^x. \qquad (*)$$

Note what happens if we put in $b=e^a$ for some real number a:

(1) On the one hand, $(*)$ gives

$$\frac{d}{dx}[(e^a)^x] = \ln(e^a)(e^a)^x = \ln(e^a)e^{ax}. \qquad (A)$$

e^x property (vi) from Monday

(2) On the other hand,

$$\frac{d}{dx}[(e^a)^x] = \frac{d}{dx}[e^{ax}] = e^{ax}\cdot\frac{d}{dx}[ax] = ae^{ax}. \qquad (B)$$

Comparing (A) and (B), we find that $\ln(e^a)e^{ax} = ae^{ax}$.
Dividing this through by e^{ax} then gives:

$$\boxed{\ln(e^a) = a} \qquad \text{(FLIP)}$$

for any real number a.

That is: "taking ln of e^a gets the a out of the exponent."

(C) Motivating example, revisited: we got to the point where we wanted to solve

$$e^{-k\tau} = 1/2 \quad \text{for } \tau.$$

Now we can! Namely, apply "ln" to both sides:

$$\ln(e^{-k\tau}) = \ln(1/2)$$

So by (FLIP),

$$-k\tau = \ln(1/2)$$

$$\tau = \frac{-\ln(1/2)}{k} = \frac{-\ln(1/2)}{1/2337} = 1619.885\ldots$$

Conclusion: radium 226 has a half-life of about 1619.885 years.

(D) Graphical interpretation.

Note that the function $y = e^x$ takes an input a to an output e^a. But (FLIP) says that $y = \ln(x)$ does the "inverse:" it takes an input e^a to an output a. In sum:

$$\boxed{\begin{aligned} e^x &: a \to e^a \\ \ln(x) &: e^a \to a \end{aligned}}$$

$y = e^x$ and $y = \ln(x)$ are <u>inverse</u> <u>functions</u>

Geometrically, swapping input with output amounts to swapping horizontal with vertical, which amounts to "flipping" (reflecting) about the line $y = x$.

SO: $y = \ln(x)$ <u>is</u> the flip of $y = e^x$ about $y = x$. Picture: