Exponential growth and decay.

(I) Exponential (unlimited) growth.

Let
$$P = P_0 e^{kt} \quad (P_0 \text{ and } k \text{ are parameters; } k \geq 0).$$

Then:

(1) $$\frac{dP}{dt} = \frac{d}{dt}[P_0 e^{kt}] = P_0 \frac{d}{dt}[e^{kt}]$$
$$= P_0 \cdot e^{kt} \cdot \frac{d}{dt}[kt] = P_0 \cdot e^{kt} \cdot k$$
$$= k \cdot \underbrace{P_0 e^{kt}}_{} = kP.$$

remember that this <u>is</u> P

(2) $$P(0) = P_0 \cdot e^{k \cdot 0} = P_0 \cdot 1 = P_0.$$

CONCLUSION:

The function $P = P_0 e^{kt}$ satisfies the exponential (or unlimited) growth initial value problem (IVP*)

$$\underbrace{\frac{dP}{dt} = kP,}_{\text{differential equation (DE)}} \qquad \underbrace{P(0) = P_0.}_{\text{initial condition (IC)}}$$

(E_g)

* IVP means: one or more DE's together with one or more IC's.

<u>Remark.</u> In (E_g), k is called the <u>per capita</u> growth rate; P_0 is the <u>initial value</u> of P.

Example 1.

A population P grows at a yearly rate equal to 0.3 times the population size. If $P = 100{,}000$ when $t=0$, find

(a) A formula for $P(t)$ (t in years);

(b) The population four years on.

Solution.

(a) We're given that
$$\frac{dP}{dt} = 0.3P, \qquad P(0) = 100{,}000.$$
So by (E_g),
$$P(t) = 100{,}000e^{0.3t}.$$
(b) By part (a),
$$P(4) = 100{,}000e^{0.3(4)} = 332{,}011.69\ldots \text{ people.}$$

(II) Exponential decay.

Reasoning as above, we find:

$$\boxed{\begin{array}{l}\text{The function } R = R_0e^{-kt} \text{ satisfies the exponential decay IVP} \\ \qquad \dfrac{dR}{dt} = -kR, \qquad R(0) = R_0.\end{array}} \qquad (E_d)$$

Here, $k>0$ is the (per unit) decay rate; R_0 is the initial value.

Example 2.

Sugar dissolves in water at a rate proportional to the amount remaining. Write an equation for the amount $S(t)$ of sugar present at time t, in terms of the decay rate k and initial amount S_0 (at $t=0$).

<u>Solution.</u>

We have $\frac{dS}{dt} = -kS$ and $S(0) = S_0$, so by (E_d),

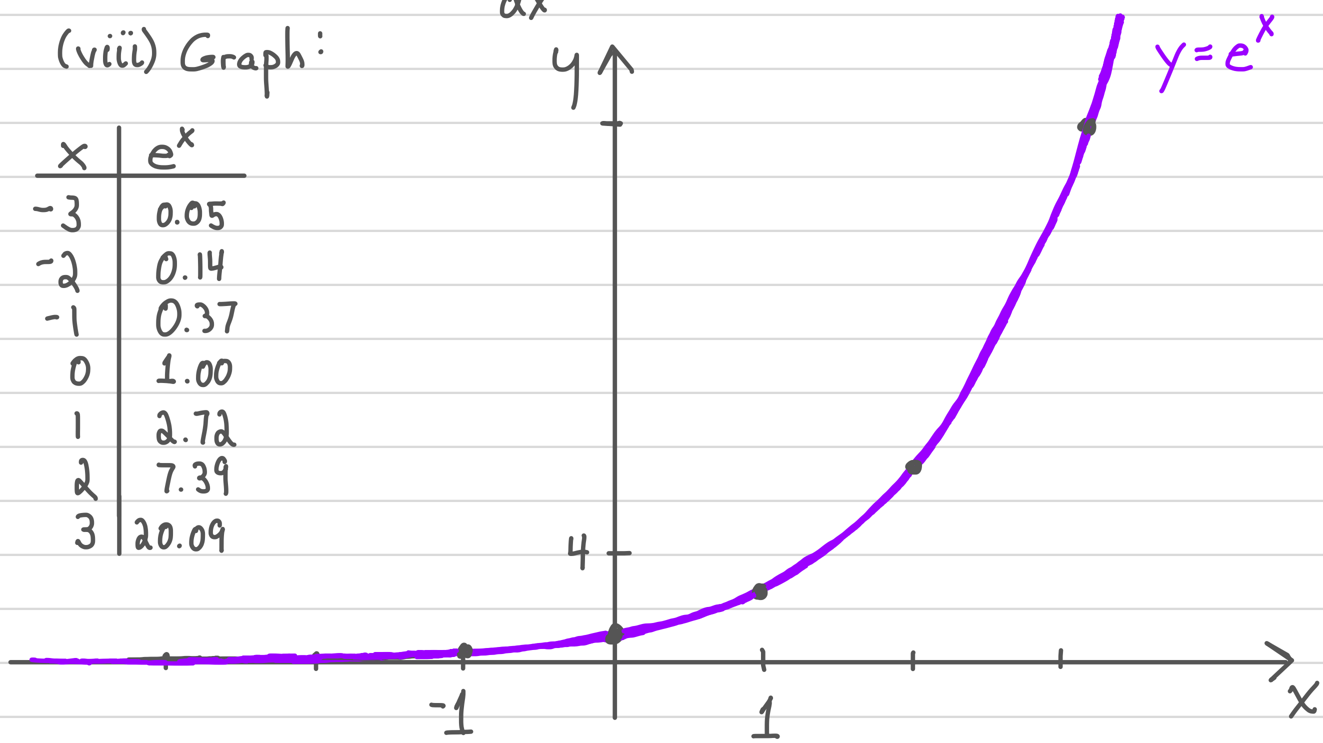
$$S(t) = S_0 e^{-kt}.$$

(III) Basic properties of $y = e^x$.

(i) $e^0 = 1$.
(ii) $e^x > 0$ for all x.
(iii) $e^{-x} = 1/e^x$ for all x.
(iv) $e^{a+b} = e^a e^b$ for all a, b.
(v) $e^{a-b} = e^a / e^b$ for all a, b.
(vi) $(e^r)^s = e^{rs}$ for all r, s.
(vii) $\frac{d}{dx}[e^x] = e^x$.

(viii) Graph:

x	e^x
-3	0.05
-2	0.14
-1	0.37
0	1.00
1	2.72
2	7.39
3	20.09



<u>Example 3.</u> Let $f(x) = e^x e^{7-x}$.

Find $f'(x)$ in two ways:
(a) Differentiate, then simplify;
(b) Simplify, then differentiate.

Solution.

(a) $f'(x) = \frac{d}{dx}\left[e^x e^{7-x}\right]$
$= e^x \frac{d}{dx}\left[e^{7-x}\right] + e^{7-x}\frac{d}{dx}\left[e^x\right]$
$= e^x e^{7-x}\frac{d}{dx}\left[7-x\right] + e^{7-x}e^x$
$= e^x e^{7-x}(-1) + e^{7-x}e^x = 0$
(the two summands cancel).

(b) $f(x) = e^x e^{7-x} = e^{x+7-x} = e^7$ (a constant),

so

$$f'(x) = \frac{d}{dx}\left[e^7\right] = 0.$$