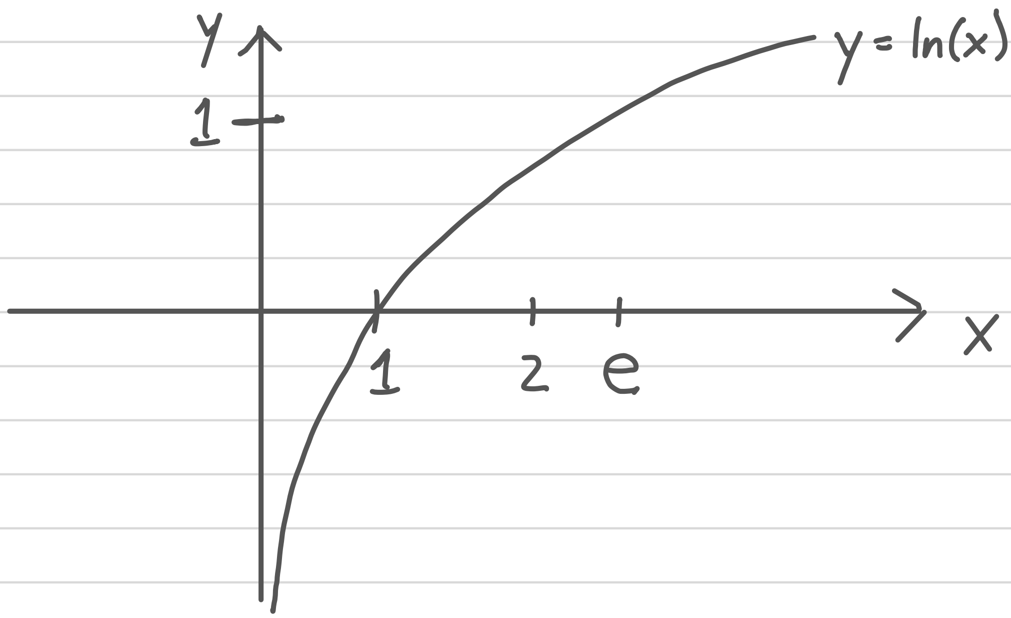
More on $\ln(x)$.

Recall: $y = e^x$ takes an input a to an output e^a. Also,

$$\boxed{\ln(e^a) = a} \qquad (\text{FLIP})$$

for all real numbers a, so $y = \ln(x)$ takes an input e^a to an output a.
Conclusion: the graph of $y = \ln(x)$ is the "flip" of $y = e^x$ about the line $y = x$.



I) Properties of $\ln(x)$.

These all come from (FLIP) and properties of e^x. For example, putting $b=0$ into (FLIP) gives $\ln(e^0) = 0$. But $e^0 = 1$, so:

(a) $\ln(1) = 0$. Some others:
(b) $\ln(e) = 1$.
(c) $\ln(1/x) = -\ln(x)$ for $x > 0$.
(d) $\ln(xy) = \ln(x) + \ln(y)$ for $x, y > 0$.
(e) $\ln(x/y) = \ln(x) - \ln(y)$ for $x, y > 0$.
(f) $\ln(r^s) = s\ln(r)$ for $r > 0$.

Example 1.
Simplify $\ln\left(\dfrac{a^{10}e^{x^2}}{b^{\sin(x)}}\right)$.

Solution. By properties (d)(e)(f), and by (FLIP),

$$\ln\left(\frac{a^{10}e^{x^2}}{b^{\sin(x)}}\right) = \ln(a^{10}e^{x^2}) - \ln(b^{\sin(x)})$$
$$= \ln(a^{10}) + \ln(e^{x^2}) - \ln(b^{\sin(x)})$$
$$= 10\ln(a) + x^2 - \sin(x)\ln(b).$$

Now note that, since $y=\ln(x)$ takes a positive number b to $\ln(b)$, its flip $y=e^x$ about $y=x$ takes $\ln(b)$ to b. That is:

$$\boxed{e^{\ln(b)}=b} \qquad \text{(FLOP)}$$

for any positive number b.

Example 2: simplify $e^{x\ln(29)+\ln(\cos(x))}$.

Solution. By (FLOP) and properties of exponentials,

$$e^{x\ln(29)+\ln(\cos(x))} = e^{x\ln(29)}e^{\ln(\cos(x))}$$
$$=(e^{\ln(29)})^x\cos(x) = 29^x\cos(x).$$

II) The derivative of $\ln(x)$.
We start with (FLOP) (with x in place of b):

$$e^{\ln(x)}=x.$$

Differentiate both sides (use the chain rule on the left):

$$e^{\ln(x)} \frac{d}{dx}[\ln(x)] = 1.$$

Now divide by $e^{\ln(x)}$:

$$\frac{d}{dx}[\ln(x)] = \frac{1}{e^{\ln(x)}} \overset{\text{by (FLOP)}}{=} \frac{1}{x}.$$

Summary:

$$\boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x}.}$$

Differentiation formula (G)

Example 3. Find

(a) $\frac{d}{dx}[\cos(1-\ln(x))]$ (b) $\frac{d}{dx}[\ln(1-\cos(x))]$ (c) $\frac{d}{dx}[\ln(e^x)]$.

Solution.

(a) $$\frac{d}{dx}[\cos(1-\ln(x))] = -\sin(1-\ln(x)) \cdot \frac{d}{dx}[1-\ln(x)]$$

$$= -\sin(1-\ln(x)) \cdot [0 - 1/x] = \frac{\sin(1-\ln(x))}{x}.$$

(b) $$\frac{d}{dx}[\ln(1-\cos(x))] = \frac{1}{1-\cos(x)} \cdot \frac{d}{dx}[1-\cos(x)] = \frac{\sin(x)}{1-\cos(x)}.$$

(c) $$\frac{d}{dx}[\ln(e^x)] = \frac{1}{e^x} \cdot \frac{d}{dx}[e^x] = \frac{1}{e^x} \cdot e^x = 1.$$

(Note: we can also simplify first:

$$\frac{d}{dx}[\ln(e^x)] \underset{\text{by (FLIP)}}{=} \frac{d}{dx}[x] = 1. \quad)$$